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MODAL ANALYSIS OF MULTI-DEGREE-OF-FREEDOM SYSTEMS WITH REPEATED FREQUENCIES – ANALYTICAL APPROACH

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The paper deals with the modal analysis of mechanical systems consisting of n identical masses connected with springs in such a way that the stiffness matrix has the form of a multiband symmetric matrix. The eigenvalue problem formulated for such systems is characterised by repeated eigenvalues to which linearly independent eigenvectors correspond. The solution to the eigenvalue problem has been found for an arbitrary, finite number of degrees of freedom for the fully coupled systems and the systems in which the masses are exclusively connected with the nearest neighbours. Depending on the property of the eigenvectors, two forms of the solution to the initial value problem have been derived. The general deliberations are illustrated with an example of the system with 10 degrees of freedom for 5 different degrees of coupling.

Key words: modal analysis, conservative systems, repeated eigenvalues

1. Introduction

Dynamic analysis of systems with repeated frequencies is interesting not only from the theoretical point of view. There are many technical problems concerning such systems, e.g. the influence of the imperfection on the mode shapes (Wei and Pierre, 1998). The examples analysed in Wang *et al.* (2005) and Pešek (1995) refer to a special type of two degrees of freedom systems possessing double frequency. It seems to be worthwhile to study multi-degree-of-freedom systems with many repeated eigenvalues possessing various multiplicity.

One of the simplest systems which possesses natural frequency of multiplicity 2 consists of three identical masses, connected with one another with the use of identical springs. Such a system has one zero frequency to which corresponds rigid-body motion and a double frequency to which corresponds harmonic motion. One should expect that systems with a larger number of degrees of freedom, consisting of identical masses and springs, will also possess repeated frequencies. It turns out that depending on the degree of coupling, the number of frequencies and their multiplicity may be different. The eigenvalue problem of such systems possesses, among others, repeated eigenvalues and mutually orthogonal eigenvectors belonging to them. Since such systems have a regular structure, there is a possibility of deriving analytical formulae for natural frequencies and modes of vibration for an arbitrary, finite number of degrees of freedom. The paper presents the way of deriving formulae for eigenvalues and eigenvectors of regular systems for various degrees of coupling as well as two forms of analytical solution to the initial value problem.

2. Fully coupled systems

Let us consider a mechanical system consisting of n identical elements of mass m, connected – each one with each one – through springs of stiffness k and, additionally, connected with the base through springs of stiffness p. The potential energy U and the kinetic energy E of the system have the form

$$U = \frac{1}{2}k\sum_{i=1}^{n}\sum_{j=i+1}^{n}(q_j - q_i)^2 + \frac{1}{2}p\sum_{i=1}^{n}q_i^2 \qquad E = \frac{1}{2}m\sum_{i=1}^{n}\dot{q}_i^2 \tag{2.1}$$

where q_i and $\dot{q}_i$ denote generalized coordinates and velocities, respectively.

2.1. Equations of motion

Equations of motion derived by means of Lagrange's equations have the form

$$\mathbf{M}\ddot{\boldsymbol{q}} + \mathbf{K}\boldsymbol{q} = \mathbf{0} \tag{2.2}$$

where the stiffness matrix is the symmetric Toeplitz matrix of the form

$$\mathbf{K} = \begin{bmatrix} (n-1)k+p & -k & \cdots & -k & -k \\ -k & (n-1)k+p & \cdots & -k & -k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -k & -k & \cdots & (n-1)k+p & -k \\ -k & -k & \cdots & -k & (n-1)k+p \end{bmatrix}_{n\times n} \tag{2.3}$$

while the inertia matrix is a scalar matrix, containing the mass m on the main diagonal.

2.2. The eigenvalue problem

Seeking a solution to Eq. (2.2) in the form

$$\boldsymbol{q} = \boldsymbol{u}\mathrm{e}^{\alpha t} \tag{2.4}$$

we obtain the algebraic eigenvalue problem in the standard form

$$\mathbf{A}\boldsymbol{u} = \lambda\boldsymbol{u} \tag{2.5}$$

where matrix $\mathbf{A}$ has the following form

$$\mathbf{A}=\begin{bmatrix} (n-1)\frac{k}{m}+\frac{p}{m} & -\frac{k}{m} & \cdots & -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & (n-1)\frac{k}{m}+\frac{p}{m} & \cdots & -\frac{k}{m} & -\frac{k}{m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{k}{m} & -\frac{k}{m} & \cdots & (n-1)\frac{k}{m}+\frac{p}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} & \cdots & -\frac{k}{m} & (n-1)\frac{k}{m}+\frac{p}{m} \end{bmatrix}_{n\times n} \tag{2.6}$$

and the unknown scalar λ is related to exponent α by the formula

$$\lambda = -\alpha^2 \tag{2.7}$$

The determinant of $\mathbf{A}$ can be written as (Bernstein, 2005)

$$\det\mathbf{A} = \frac{p}{m}\Big(n\frac{k}{m}+\frac{p}{m}\Big)^{n-1} \tag{2.8}$$

Since the determinant of the matrix is equal to the product of its eigenvalues, one can suppose that eigenvalues of $\mathbf{A}$ will have the form

$$\begin{cases} \lambda_1 = \dfrac{p}{m} \\ \lambda_j = n\dfrac{k}{m}+\dfrac{p}{m} \qquad j=2,3,\dots,n \end{cases} \tag{2.9}$$

As can be seen λ_1 is a single eigenvalue while λ_2 is repeated $n-1$ times. Putting the eigenvalue λ_1 into Eq. (2.5) we obtain a set of equations with respect to the elements of the eigenvector belonging to λ_1 in the form

$$\begin{bmatrix} n-1 & -1 & \cdots & -1 & -1 \\ -1 & n-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & n-1 & -1 \\ -1 & -1 & \cdots & -1 & n-1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \mathbf{0} \tag{2.10}$$

The rank of the coefficient matrix is equal to $n-1$ and, as it is easy to notice, the solution to Eq. (2.10) has the form

$$\boldsymbol{u}_1 = [1, 1, \ldots, 1, 1]^\top \tag{2.11}$$

The eigenvectors corresponding to the eigenvalue λ_2 should fulfill the following set of equations

$$\begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \mathbf{0} \tag{2.12}$$

Now, the rank of the coefficient matrix is equal to 1, which means that it is possible to choose freely $n-1$ components of the vector $\boldsymbol{u}$. The simplest choice consists in imposing the values: 1 and -1 on two components of $\boldsymbol{u}_j$, $j = 2, 3 \ldots, n$ while putting others as zero. The value -1 can be assigned to the first component, whereas the value 1 should have a changeable position – from the second one to the last one. The matrix $\mathbf{U}$ created from such eigenvectors has the form

$$\mathbf{U} = \begin{bmatrix} 1 & -1 & \cdots & -1 & -1 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n \times n} \tag{2.13}$$

The eigenvectors $\boldsymbol{u}_j$, $j = 2, 3 \ldots, n$ constitute a set of $n-1$ linearly independent vectors, so the geometrical multiplicity of the eigenvalue λ_2 is equal to its algebraic multiplicity, therefore the matrix $\mathbf{U}$ diagonalizes the matrix $\mathbf{A}$, i.e.

$$\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{2.14}$$

Relation (2.14) confirms the accuracy of Eqs. (2.9) and (2.13) which describe the eigenvalues and eigenvectors of matrix $\mathbf{A}$.

2.3. Solution of the initial value problem in the base of linearly independent vectors

To every (different from zero) eigenvalue λ_j there correspond, according to Eq. (2.7), two imaginary mutually coupled values of the exponent α

$$\alpha_j = \sqrt{\lambda_j}\,\mathrm{i} \qquad \alpha_j^* = -\sqrt{\lambda_j}\,\mathrm{i} \qquad \mathrm{i} = \sqrt{-1} \tag{2.15}$$

to which, in turn, corresponds a solution to Eq. (2.2) in the form

$$\begin{aligned} \boldsymbol{q}_j &= (A_j \mathrm{e}^{\alpha_j t} + A_j^* \mathrm{e}^{\alpha_j^* t})\boldsymbol{u}_j = \Big[\frac{1}{2}(C_j - \mathrm{i}S_j)\mathrm{e}^{\sqrt{\lambda_j}\mathrm{i}t} + \frac{1}{2}(C_j + \mathrm{i}S_j)\mathrm{e}^{-\sqrt{\lambda_j}\mathrm{i}t}\Big] \\ &= (C_j \cos\sqrt{\lambda_j}t + S_j \sin\sqrt{\lambda_j}\, t)\boldsymbol{u}_j \end{aligned} \tag{2.16}$$

Due to the fact that the eigenvectors $\boldsymbol{u}_j$, $j = 1, 2, \ldots, n$ are linearly independent, the complete solution will have the form

$$\boldsymbol{q} = \sum_{j=1}^{n} \boldsymbol{q}_j = \sum_{j=1}^{n} (C_j \cos\omega_j t + S_j \sin\omega_j t)\boldsymbol{u}_j \tag{2.17}$$

where ω_j denote natural frequencies of the system, related to the eigenvalues by the formula

$$\omega_j = \sqrt{\lambda_j} \qquad j = 1, 2, \ldots, n \tag{2.18}$$

In the case of the lack of the coupling with the base $(p = 0)$, the first eigenvalue equals 0, so Eq. (2.17) will now have the form

$$\boldsymbol{q} = (C_1 + tS_1)\boldsymbol{u}_1 + \sum_{j=2}^{n} (C_j \cos\omega_j t + S_j \sin\omega_j t)\boldsymbol{u}_j \tag{2.19}$$

The constants C_j and S_j should be found out from the initial conditions

$$\boldsymbol{q}(0) = \boldsymbol{q}_0 \qquad \dot{\boldsymbol{q}}(0) = \boldsymbol{p}_0 \tag{2.20}$$

Equation (2.20) constitutes a set of $2n$ linear algebraic equations.

2.4. Solution of the initial value problem in the base of orthogonal vectors

The eigenvectors $\boldsymbol{u}_j$, $j = 2, 3, \ldots, n$ are orthogonal with respect to the eigenvector $\boldsymbol{u}_1$ but they are not mutually orthogonal. Thus they cannot be orthogonal with respect to the scalar matrix either. For this reason, there is a need to solve the set of Eq. (2.20) to determine the constants C_j and S_j. It is possible to avoid this necessity using Gram-Schmidt orthogonalization process (Meirovitch, 1997) that renders the independent vectors orthogonal. Next, normalizing the orthogonal vectors to be orthonormal with respect to the matrix $\mathbf{M}$, we receive vectors $\boldsymbol{n}_j$, $j = 1, 2, \dots, n$. The elements of matrix $\mathbf{N}$, composed of the vectors $\boldsymbol{n}_j$, are described by formulae

$$
\begin{aligned}
n_{i1} &= \frac{1}{\sqrt{mn}} \qquad\qquad i = 1, 2, \dots, n \\
n_{ij} &= \begin{cases} \dfrac{1}{\sqrt{m}} \dfrac{-1}{\sqrt{j(j-1)}} & i < j \\ \dfrac{1}{\sqrt{m}} \dfrac{j-1}{\sqrt{j(j-1)}} & i = j \qquad\qquad j = 2, 3, \dots, n \\ 0 & j < i \leqslant n \end{cases}
\end{aligned}
\tag{2.21}
$$

The matrix $\mathbf{N}$ fulfils the following relations

$$
\begin{aligned}
&\mathbf{N}^\top \mathbf{M} \mathbf{N} = \mathbf{I} \qquad\qquad \mathbf{N}^\top \mathbf{K} \mathbf{N} = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \\
&\mathbf{N}^{-1} \mathbf{A} \mathbf{N} = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)
\end{aligned}
\tag{2.22}
$$

where $\mathbf{I}$ denotes the identity matrix. As can be seen, the matrix $\mathbf{N}$, alike matrix $\mathbf{U}$, is the similarity transformation matrix of the matrix $\mathbf{A}$.

To uncouple the equations of motion, we should introduce the vector $\boldsymbol{x}$ of natural coordinates by means of the linear transformation

$$
\boldsymbol{q} = \mathbf{N}\boldsymbol{x} \tag{2.23}
$$

Substituting Eq. (2.23) into Eq. (2.2) and premultiplying the result by $\mathbf{N}^\top$, we obtain an uncoupled set of equations in the form

$$
\ddot{\boldsymbol{x}} + \boldsymbol{\Lambda}\boldsymbol{x} = \mathbf{0} \tag{2.24}
$$

where $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Making use of linear transformation (2.23), one can easily determine the initial conditions for the natural coordinates in the form

$$
\boldsymbol{x}_0 = \mathbf{N}^\top \mathbf{M} \boldsymbol{q}_0 \qquad\qquad \boldsymbol{v}_0 = \mathbf{N}^\top \mathbf{M} \boldsymbol{p}_0 \tag{2.25}
$$

The solution to Eq. (2.24) for j-th natural coordinate $(\lambda_j \neq 0)$ can be expressed as

$$
x_j = \boldsymbol{n}_j^\top \mathbf{M} \Big(\boldsymbol{q}_0 \cos\omega_j t + \frac{\boldsymbol{p}_0}{\omega_j} \sin\omega_j t \Big) \tag{2.26}
$$

whereas the complete solution becomes

$$\boldsymbol{q} = \mathbf{N}\boldsymbol{x} = \sum_{j=1}^{n} x_j \boldsymbol{n}_j = \sum_{j=1}^{n} \boldsymbol{n}_j^\top \mathbf{M}\Big(\boldsymbol{q}_0 \cos\omega_j t + \frac{\boldsymbol{p}_0}{\omega_j}\sin\omega_j t\Big)\boldsymbol{n}_j \tag{2.27}$$

In the case of the lack of the coupling with the base $(p = 0)$, solution (2.27) will have the form

$$\boldsymbol{q} = \boldsymbol{n}_1^\top \mathbf{M}(\boldsymbol{q}_0 + t\boldsymbol{p}_0)\boldsymbol{n}_1 + \sum_{j=2}^{n} \boldsymbol{n}_j^\top \mathbf{M}\Big(\boldsymbol{q}_0 \cos\omega_j t + \frac{\boldsymbol{p}_0}{\omega_j}\sin\omega_j t\Big)\boldsymbol{n}_j \tag{2.28}$$

Thanks to the use of the orthonormal vectors $\boldsymbol{n}_j$, the solutions to Eq. (2.2) in the form (2.27) or (2.28) do not require any set of algebraic equations to be solved.

3. Systems not fully coupled

In the case when each individual element of the system is not coupled with all remaining ones, the stiffness matrix $\mathbf{K}$ will have subdiagonals containing zeroes. The number of zero subdiagonals depends on the total number of masses (n) and the number of masses connected with individual element (s). Assuming the same number of neighbouring masses connected with every element on both their sides, the stiffness matrix will have the form of a symmetrical multiband matrix. For example, for $n = 8$ and $s = 4$ the stiffness matrix has the form

$$\mathbf{K} = \begin{bmatrix} 4k+p & -k & -k & 0 & 0 & 0 & -k & -k \\ -k & 4k+p & -k & -k & 0 & 0 & 0 & -k \\ -k & -k & 4k+p & -k & -k & 0 & 0 & 0 \\ 0 & -k & -k & 4k+p & -k & -k & 0 & 0 \\ 0 & 0 & -k & -k & 4k+p & -k & -k & 0 \\ 0 & 0 & 0 & -k & -k & 4k+p & -k & -k \\ -k & 0 & 0 & 0 & -k & -k & 4k+p & -k \\ -k & -k & 0 & 0 & 0 & -k & -k & 4k+p \end{bmatrix} \tag{3.1}$$

In such systems, there are repeated eigenvalues as well as single ones. The formulae for eigenvalues and eigenvectors are not as simple as in fully coupled systems. However, the described procedure for obtaining the analytical solution is still the same.

4. Systems with the lowest degree of coupling

The systems with the lowest degree of coupling consist of identical masses connected exclusively with the nearest neighbours $(s = 2)$ where the first mass is connected with the second one and the last one. The schematic diagram of such a system (not coupled with the base) is presented in Fig. 1.

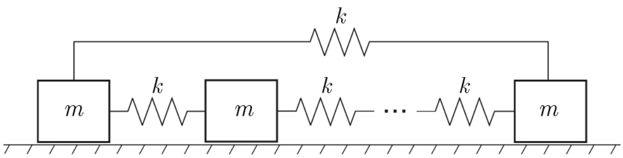

Fig. 1. Schematic diagram of the system with the lowest degree of coupling

The system shown in Fig. 1 has its continuous counterpart in the form of an unrestrained prismatic bar with the ends connected with each other by a rigid weightless link. Making use of the dispersion formula, which relates the natural frequency and the length of the wave, it is possible to derive a formula for natural frequencies of the system in the form (Palej and Goik, 2003)

$$\omega_j = 2\sqrt{\frac{k}{m}} \sin \frac{\pi(j-1)}{n} \qquad j = 1, 2, \dots, n \tag{4.1}$$

The squares of natural frequencies (4.1) determine, according to Eq. (2.18), the eigenvalues of the matrix

$$\mathbf{A} = \frac{k}{m} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}_{n \times n} \tag{4.2}$$

In the case when the masses are also coupled with the base $(p \neq 0)$, we obtain the eigenvalue problem in the standard form of the matrix $\mathbf{B}$

$$\mathbf{B} = \mathbf{A} + \frac{p}{m}\mathbf{I} \tag{4.3}$$

The addition of p/m to the diagonal elements of $\mathbf{A}$ produces a shift in the eigenvalues by the same constant, so the eigenvalues of the matrix $\mathbf{B}$ take the form

$$\lambda_j = 4\frac{k}{m}\sin^2\frac{\pi(j-1)}{n} + \frac{p}{m} \qquad j = 1,2,\dots,n \tag{4.4}$$

It appears from Eq. (4.4) that the smallest eigenvalue λ_1 is single, while the remaining eigenvalues, in the case when n is odd, are double. In the case when n is even, the smallest eigenvalue λ_1 as well as the biggest one $\lambda_{n/2+1}$ are single, while the remaining eigenvalues are double. It should be pointed out that Eqs. (2.9) and (4.4) determine the same eigenvalues for the system consisting of three masses ($n = 3$, $s = 2$). The way of coupling in such a system fulfils the requirements characteristic of fully coupled systems and the systems with the lowest degree of coupling.

The eigenvectors $\boldsymbol{u}_j$ of matrix $\mathbf{A}$ form the modal matrix $\mathbf{U}$, which has elements of the form (Palej and Goik, 2003)

$$u_{ij} = \begin{cases} \cos\dfrac{2\pi(j-1)i}{n} & j \leqslant j_{sep} \\ \sin\dfrac{2\pi(j-1)i}{n} & j_{sep} < j \leqslant n \end{cases} \qquad i,j = 1,2,\dots,n \tag{4.5}$$

where

$$j_{sep} = \begin{cases} \dfrac{n+1}{2} & n \ \text{odd} \\ \dfrac{n}{2}+1 & n \ \text{even} \end{cases} \tag{4.6}$$

The eigenvectors $\boldsymbol{u}_j$ are mutually orthogonal, therefore they are orthogonal with respect to any scalar matrix too. Normalizing them to be orthonormal with respect to the matrix $\mathbf{M}$, we receive vectors $\boldsymbol{n}_j$, $j = 1,2,\dots,n$ which are required to make use of solution (2.27) or (2.28).

5. Illustrative example

Eigenvalues of fully coupled systems and systems with the lowest degree of coupling are determined in an analytical way for an arbitrary, finite number of masses by formulae (2.9) and (4.4), respectively. In the first case, the second eigenvalue is repeated $n-1$ times, while in the second case, the multiplicity of individual eigenvalues does not exceed 2. In the case of not fully coupled systems, the multiplicity of eigenvalues can be different. The spectra of natural frequencies of the system consisting of 10 masses for 5 different degrees of coupling are presented in Fig. 2.

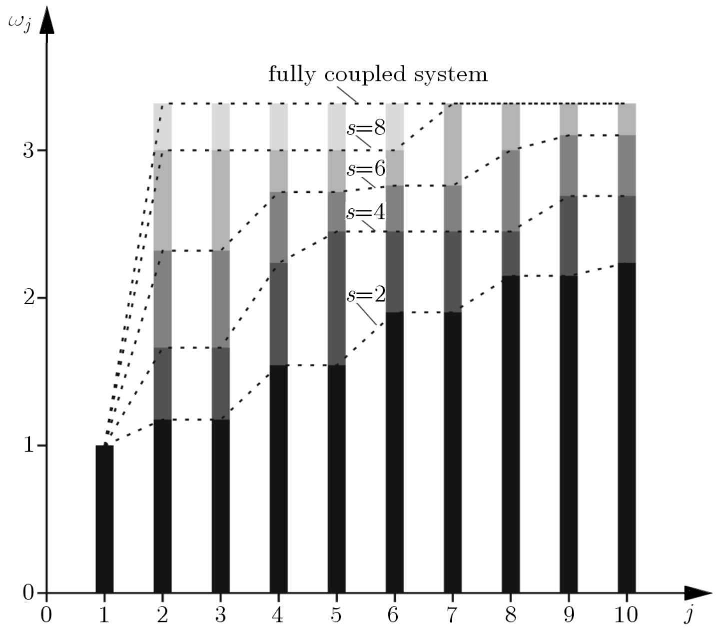

Fig. 2. Spectra of natural frequencies of the regular system consisting of 10 masses for 5 different degree of coupling, for $m = 1\,\mathrm{kg}$ and $k = p = 1\,\mathrm{N/m}$

The individual spectra were diversified by a different degree of greyness. Additionally, the dotted lines join the peaks of stripes belonging to one spectrum. The horizontal segments of these lines indicate the repeated frequencies. As can be seen in Fig. 2, the lowest frequency is the same for all types of coupling and the multiplicity of repeated frequency may vary between 2 and 9.

6. Conclusions

For undamped natural systems possessing distinct eigenvalues, to every eigenvalue corresponds one unique eigenvector. The eigenvalues determine, by a suitable formula, natural frequencies while the eigenvectors determine directly the modes of vibration. Repeated eigenvalues can appear in discrete systems, consisting of identical masses and springs – arranged in such a way that every mass is constrained in the same manner. In such systems, to each repeated eigenvalue there corresponds a set of linearly independent eigenvectors which determine the modes of vibration. Since any linear combination of linearly independent eigenvectors corresponding to the repeated eigenvalue is also an eigenvector, therefore, an infinite number of modes of vibration correspond to the repeated eigenvalue. It means that such systems can execute harmonic motion with natural frequency corresponding to the repeated eigenvalue in infinite ways. To perform such motion, it is enough to set a vector of initial displacements $\boldsymbol{q}_0$ and/or initial velocities $\boldsymbol{p}_0$ as a linear combination of the eigenvectors corresponding to the repeated eigenvalue.

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Analiza modalna układów o wielu stopniach swobody z wielokrotnymi częstościami drgań – ujęcie analityczne

Streszczenie

Praca dotyczy analizy modalnej układów mechanicznych, zbudowanych z n identycznych mas połączonych sprężynami w taki sposób, by macierz sztywności miała budowę wielopasmowej macierzy symetrycznej. Zagadnienie własne sformułowane dla tych układów charakteryzuje się wielokrotnymi wartościami własnymi, którym odpowiadają układy liniowo niezależnych wektorów własnych. W pracy podano analityczne rozwiązanie zagadnienia własnego dla dowolnej, skończonej liczby stopni swobody, dla układu w pełni sprzężonego i układu, w którym każda masa połączona jest wyłącznie z dwiema sąsiednimi. W zależności od własności wektorów własnych podano dwie postacie rozwiązania zagadnienia początkowego. Rozważania teoretyczne zilustrowano przykładem układu o 10 stopniach swobody, dla 5 różnych stopni sprzężenia.

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