

TREFFTZ FUNCTIONS FOR A PLATE VIBRATION PROBLEM

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The paper presents a new method to obtain an approximate solution to plate vibrations problems. The problem is described by a partial differential equation of fourth order. The key idea of the presented approach is to find polynomials (solving functions) that satisfy the considered differential equation identically. In this sense, it is a variant of the Trefftz method. The method is addressed to differential equations in a finite domain. The approach proposed here has some advantages. The first is that the approximate solution (a linear combination of the solving functions) satisfies the equation identically. Secondly, the method is flexible in terms of given boundary and initial conditions (discrete, missing). Thirdly, the solving functions can be used as base functions for several variants of the Finite Element Method. In this case, the approximation is good even for relatively large elements. It means that the approach is suitable for inverse problems. The formulas for solving functions and their derivatives for the plate vibration equation are obtained. The convergence of the method is proved and numerical examples are included.

Key words: plate vibration, Trefftz functions, FEM

1. Introduction

There are a lot of methods for solving partial differential equations. They may be divided into three groups. The first group contains the so-called analytical methods. It means that the solution exactly satisfies both the equation and all the given conditions (e.g. integral-transformation, separation-of-variables, Green's function and so on). Unfortunately, these methods are not always useful for numerical calculations (some problems with numerical convergence of series). Moreover, the shape of the body should be relatively simple. The second

group contains methods such as the Finite Element Method or the Boundary Element Method. Then the solution approximately satisfies both the equation and all the given conditions and the shape does not have to be simple. In the third group, the solution exactly satisfies the equation and approximately all given conditions (the Trefftz method (Trefftz, 1926)). The method presented here belongs to the third group. The key idea of the method is to find functions (polynomials) satisfying the given differential equation (solving functions) to be fitted to the governing initial and boundary conditions.

The Trefftz functions method for linear partial differential equations has been developed by Herrera and Sabina (1978), Jirousek (1978), Jirousek *et al.* (1997), Kołodziej and Zieliński (2009), Kupradze (1989), Zieliński and Zienkiewicz (1985). Especially the monograph Kołodziej and Zieliński (2009) contains a large survey of the T-functions method and their applications to engineering problems. All the above mentioned authors considered equations without time variable or the time was discretized. A little bit different approach towards using Trefftz functions (the solution depends in continuous way on time) was first described in the paper by Rosenbloom and Widder (1956), where it was applied to solving one-dimensional heat conduction equation. Although his approach has been developed only recently, it has rich literature. In the papers by Ciałkowski *et al.* (1999a,b), Futakiewicz (1999), Futakiewicz *et al.* (1999); Futakiewicz and Hożejowski (1998a,b), Hożejowski (1999), Yano *et al.* (1983) the authors described heat functions in different coordinate systems and for direct and inverse heat conduction problems. The making use of heat polynomials as a new type of finite-element base functions is described in Ciałkowski (1999). The paper by Ciałkowski and Frąckowiak (2000) deals with numerous cases, involving other differential equations such as the Laplace, Poisson and Helmholtz ones. Solving functions for the wave equation are presented in Ciałkowski (2003), Ciałkowski and Frąckowiak (2000, 2003, 2004), Ciałkowski and Jarosławski (2003), Maciąg (2004, 2005) Maciąg and Wauer (2005a,b). Applications of wave polynomials for elasticity problems are shown in Maciąg (2007b). Solving polynomials for beam vibration problems are described in Al-Khatib *et al.* (2008).

The properties of Taylor series are very useful in the method of solving functions

$$\begin{aligned}
 f(x + \Delta x, y + \Delta y, t + \Delta t) &= f(x, y, t) + \\
 &+ \frac{df}{1!} + \frac{d^2f}{2!} + \cdots + \frac{d^N f}{N!} + R_{N+1}
 \end{aligned}
 \tag{1.1}$$

where

$$d^n f = \left(\frac{\partial}{\partial x} \Delta x + \frac{\partial}{\partial y} \Delta y + \frac{\partial}{\partial t} \Delta t \right)^n f$$

and R_{N+1} is the remainder term.

Based on this, in Section 2, plate polynomials and their properties are considered. In Section 3, the method of solving functions is described. Section 4 contains some remarks on convergence of approximation. Section 5 describes how to use plate polynomials in the polar coordinate system. In Section 6, a test example is discussed. Here two ways of solving are shown – in the whole domain and nodeless FEM. Concluding remarks are given in Section 7.

2. Solving polynomials for the plate vibration equation

There are two ways to obtain solving polynomials for the plate vibration equation (let us name them "plate polynomials"). The first is to use a "generating function". The second is to expand the function satisfying the equation into power series and reduct some components of the expansion with the use of this equation. Both methods are equivalent and lead to the same polynomials.

2.1. Generating function

Let us consider the equation which describes vibrations of plates

$$\begin{aligned} \nabla^2(D\nabla^2 u) + (1 - \nu) \left[2 \frac{\partial^2}{\partial x \partial y} \left(D \frac{\partial^2 u}{\partial x \partial y} \right) - \frac{\partial^2}{\partial x^2} \left(D \frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial^2}{\partial y^2} \left(D \frac{\partial^2 u}{\partial x^2} \right) \right] = \\ = Q(x, y, \tau) - \rho \frac{\partial^2 u}{\partial \tau^2} \end{aligned} \quad (2.1)$$

here ν is the Poisson ratio, Q – load (inhomogeneity), D – stiffness of the plate. If the plate is homogeneous and has constant thickness then $D(x, y) = \text{const}$, and equation (2.1) has form

$$D\nabla^4 u = Q(x, y, \tau) - \rho \frac{\partial^2 u}{\partial \tau^2} \quad (2.2)$$

If we assume further dimensionless time $t = \tau \sqrt{D/\rho}$, then we get a dimensionless equation

$$\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^2 u}{\partial t^2} = Q(x, y, t) \quad (2.3)$$

We use the dimensionless time for the only reason of convenience – formulas for Trefftz functions obtained below are simpler. First we consider the homogeneous equation (inhomogeneity is considered further on)

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^2 u}{\partial t^2} = 0 \quad (2.4)$$

The function

$$g = e^{ax+by+i(a^2+b^2)t} \quad (2.5)$$

satisfying equation (2.4) is called a generating function. The power series expansion (1.1) for (2.5) is

$$g = \sum_{n=0}^{\infty} \sum_{k=0}^n R_{nk} a^{n-k} b^k = \sum_{n=0}^{\infty} \sum_{k=0}^n (P_{nk} + i(a^2 + b^2)Q_{nk}) a^{n-k} b^k \quad (2.6)$$

where $P_{nk}(x, y, t)$, $Q_{nk}(x, y, t)$ are polynomials satisfying equation (2.4) (function g satisfies equation (2.4) for each a and b). E.g.

$$\begin{array}{ll} P_{00}(x, y, t) = 1 & Q_{00}(x, y, t) = t \\ P_{10}(x, y, t) = x & Q_{10}(x, y, t) = xt \\ P_{11}(x, y, t) = y & Q_{11}(x, y, t) = yt \\ P_{20}(x, y, t) = \frac{x^2}{2} & Q_{20}(x, y, t) = \frac{x^2 t}{2} \\ P_{21}(x, y, t) = xy & Q_{21}(x, y, t) = xyt \\ P_{22}(x, y, t) = \frac{y^2}{2} & Q_{22}(x, y, t) = \frac{y^2 t}{2} \\ P_{30}(x, y, t) = \frac{x^3}{6} & Q_{30}(x, y, t) = \frac{x^3 t}{6} \\ P_{31}(x, y, t) = \frac{x^2 y}{2} & Q_{31}(x, y, t) = \frac{x^2 yt}{2} \\ P_{32}(x, y, t) = \frac{xy^2}{2} & Q_{32}(x, y, t) = \frac{xy^2 t}{2} \\ P_{33}(x, y, t) = \frac{y^3}{6} & Q_{33}(x, y, t) = \frac{y^3 t}{6} \\ \dots & \end{array} \quad (2.7)$$

Obtaining formulas for solving polynomials from (2.6) is inconvenient. Therefore, recurrent formulas for these polynomials and their derivatives are further shown.

2.2. Partial derivatives of the solving polynomials

Recurrent formulas are very useful in numerical calculations. The procedures to obtain space and time derivatives are similar. Therefore, we show the way to obtain derivative relative to variable x and final formulas for variables y and t . To obtain these formulas, we differentiate (2.5) with respect to suitable variable

$$\frac{\partial g}{\partial x} = ag = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{\partial R_{nk}}{\partial x} a^{n-k} b^k$$

Hence

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} R_{(n-1)k} a^{n-k} b^k = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{\partial R_{nk}}{\partial x} a^{n-k} b^k$$

and

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} R_{(n-1)k} a^{n-k} b^k = \frac{\partial R_{00}}{\partial x} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{\partial R_{nk}}{\partial x} a^{n-k} b^k + \sum_{n=1}^{\infty} \frac{\partial R_{nn}}{\partial x} b^n$$

Hence

$$\frac{\partial R_{00}}{\partial x} = \frac{\partial R_{nn}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial R_{nk}}{\partial x} = R_{(n-1)k}$$

so that finally

$$\begin{aligned} \frac{\partial P_{00}}{\partial x} = \frac{\partial Q_{00}}{\partial x} = \frac{\partial P_{nn}}{\partial x} = \frac{\partial Q_{nn}}{\partial x} &= 0 \\ \frac{\partial P_{nk}}{\partial x} = P_{(n-1)k} \quad \frac{\partial Q_{nk}}{\partial x} = Q_{(n-1)k} &\begin{cases} n > 0 \\ k = 0, 1, \dots, n-1 \end{cases} \end{aligned} \tag{2.8}$$

Similarly we have

$$\begin{aligned} \frac{\partial P_{00}}{\partial y} = \frac{\partial Q_{00}}{\partial y} = \frac{\partial P_{n0}}{\partial y} = \frac{\partial Q_{n0}}{\partial y} &= 0 \\ \frac{\partial P_{nk}}{\partial y} = P_{(n-1)(k-1)} \quad \frac{\partial Q_{nk}}{\partial y} = Q_{(n-1)(k-1)} &\begin{cases} n > 0 \\ k = 0, 1, \dots, n \end{cases} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \frac{\partial Q_{nk}}{\partial t} &= P_{nk} \\ \frac{\partial P_{nk}}{\partial t} &= 0 \quad n = 0, 1, 2, 3, \quad k = 0, \dots, n \\ \frac{\partial P_{n0}}{\partial t} &= -Q_{(n-4)0} \quad \frac{\partial P_{nn}}{\partial t} = -Q_{(n-4)(n-4)} \end{aligned}$$

$$\begin{aligned} \frac{\partial P_{n(n-1)}}{\partial t} &= -Q_{(n-4)(n-5)} & (2.10) \\ \frac{\partial P_{n(n-2)}}{\partial t} &= -2Q_{(n-4)(n-4)} - Q_{(n-4)(n-6)} \\ \frac{\partial P_{n(n-3)}}{\partial t} &= -2Q_{(n-4)(n-5)} - Q_{(n-4)(n-7)} \\ \frac{\partial P_{nk}}{\partial t} &= -Q_{(n-4)k} - 2Q_{(n-4)(k-2)} - Q_{(n-4)(k-4)} \quad \begin{cases} n > 3 \\ k = 0, 1, \dots, n-4 \end{cases} \end{aligned}$$

In formulas, (2.10), of course we put zero instead of the polynomial in question in the right-hand side if any of its subscripts takes a negative value.

2.3. Recurrent formulas for the solving polynomials

Theorem 1 enables one to get all plate polynomials P_{nk} and Q_{nk} .

Theorem 1. *Let $(P_{00} = 1)$ and $(Q_{00} = t)$. Then, the polynomials*

$$P_{n1} = yP_{(n-1)0} \quad Q_{n1} = yQ_{(n-1)0} \quad (2.11)$$

$$P_{n(n-1)} = xP_{(n-1)(n-1)} \quad Q_{n(n-1)} = xQ_{(n-1)(n-1)} \quad (2.12)$$

$$P_{(n+2)0} = \frac{2x(n+1)P_{(n+1)0} - x^2P_{n0} - 4t^2P_{(n-2)0} - 2tQ_{(n-2)0}}{(n+1)(n+2)} \quad (2.13)$$

$$Q_{n0} = \frac{2x(n+1)Q_{(n-1)0} - x^2Q_{(n-2)0} - 4t^2Q_{(n-4)0} + 2tP_{n0}}{(n+1)(n+2)} \quad (2.14)$$

and for $1 \leq k \leq n-1$

$$P_{n(k+1)} = \frac{(n-k+1)P_{n(k-1)} - xP_{(n-1)(k-1)} + yP_{(n-1)k}}{k+1} \quad (2.15)$$

$$Q_{n(k+1)} = \frac{(n-k+1)Q_{n(k-1)} - xQ_{(n-1)(k-1)} + yQ_{(n-1)k}}{k+1} \quad (2.16)$$

satisfy equation (2.4).

Proof. The proof of Theorem 1 is based on putting the considered polynomial into equation (2.4) and on using suitable formula from (2.8)-(2.10) for derivatives. For example, we proof formula (2.16). For acceptable n and k we have

$$\begin{aligned} & \frac{\partial^4 P_{n(k+1)}}{\partial x^4} + 2 \frac{\partial^4 P_{n(k+1)}}{\partial x^2 \partial y^2} + \frac{\partial^4 P_{n(k+1)}}{\partial y^4} + \frac{\partial^2 P_{n(k+1)}}{\partial t^2} = \\ & = \frac{1}{k+1} \left[(n-k+1)P_{(n-4)(k-1)} - xP_{(n-5)(k-1)} + yP_{(n-5)k} + \right. \\ & + 2(n-k+1)P_{(n-4)(k-3)} - 2xP_{(n-5)(k-3)} + 2yP_{(n-5)(k-2)} + \\ & + (n-k+1)P_{(n-4)(k-5)} - xP_{(n-5)(k-5)} + yP_{(n-5)(k-4)} + \\ & + (n-k+1)(-P_{(n-4)(k-1)} - 2P_{(n-4)(k-3)} - P_{(n-4)(k-5)}) + \\ & - x(-P_{(n-5)(k-1)} - 2P_{(n-5)(k-3)} - P_{(n-5)(k-5)}) + \\ & \left. + y(-P_{(n-5)k} - 2P_{(n-5)(k-2)} - P_{(n-5)(k-4)}) \right] = 0 \end{aligned}$$

This proves relation (2.16). The proofs for other cases are similar.

2.4. Usage of the plate vibration equation in Taylor series

Similarly as for other equations (Ciałkowski and Frąckowiak, 2000; Maciąg, 2005; Maciąg and Wauer, 2005a), the plate polynomials can be obtained using Taylor series (1.1) for the function u . Let the function $u(x, y, t)$ satisfy equation (2.4) with given initial and boundary conditions. We assume that $w \in C^{2N+1}$ in the neighborhood of (x_0, y_0, t_0) . Let $\hat{x} = x - x_0$, $\hat{y} = y - y_0$, $\hat{t} = t - t_0$. Then the function u can be expanded into power series. Further in the expansion of the function u , some derivatives can be eliminated by making use of the plate vibration equation as follows

$$\frac{\partial^4 u}{\partial x^4} = -2 \frac{\partial^4 u}{\partial x^2 \partial y^2} - \frac{\partial^4 u}{\partial y^4} - \frac{\partial^2 u}{\partial t^2} \tag{2.17}$$

or

$$\frac{\partial^4 u}{\partial y^4} = -\frac{\partial^4 u}{\partial x^4} - 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} - \frac{\partial^2 u}{\partial t^2} \tag{2.18}$$

or

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4} - 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} - \frac{\partial^4 u}{\partial y^4} \tag{2.19}$$

For example, after using the substitution according to formula (2.19) in the expansion, the coefficient by the derivative $\partial^4 u / \partial x^4$ will be $(\hat{x}^4 / 4!) - (\hat{t}^2 / 2!)$. It is the plate polynomial. Other plate polynomials we get as coefficients by successive derivatives in the expansion. Simple but a little bit onerous calculations show that this procedure leads to the same plate polynomials as obtained by using formulas in Theorem 1. This way of obtaining the plate polynomials is very important. It will be used to prove the convergence of the method of solving functions (see Section 4).

2.5. Inhomogeneity

For the non-homogeneous equation

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^2 u}{\partial t^2} = Q(x, y, t) \quad (2.20)$$

the particular solution has to be known. It is easy to calculate it if the inhomogeneity $Q(x, y, t)$ can be expanded into Taylor series. Then it is sufficient to know how to calculate the inverse operator L^{-1} for monomials, where $L = \nabla^4 + \partial^2/\partial t^2$. Denote $Z_{klm} = L^{-1}(x^k y^l t^m)$. It is easy to prove that we have four forms of $Z(x^k y^l t^m)$

$$\begin{aligned} Z_{klm}^1 &= \frac{1}{(k+4)(k+3)(k+2)(k+1)} \cdot \\ &\cdot \left[x^{k+4} y^l t^m - 2(k+4)(k+3)l(l-1)Z_{(k+2)(l-2)m} + \right. \\ &\left. -l(l-1)(l-2)(l-3)Z_{(k+4)(l-4)m} - m(m-1)Z_{(k+4)l(m-2)} \right] \\ Z_{klm}^2 &= \frac{1}{2(k+2)(k+1)(l+2)(l+1)} \cdot \\ &\cdot \left[x^{k+2} y^{l+2} t^m - (k+2)(k+1)k(k-1)Z_{(k-2)(l+2)m} + \right. \\ &\left. -(l+2)(l+1)l(l-1)Z_{(k+2)(l-2)m} - m(m-1)Z_{(k+2)(l+2)(m-2)} \right] \\ Z_{klm}^3 &= \frac{1}{(l+4)(l+3)(l+2)(l+1)} \cdot \\ &\cdot \left[x^k y^{l+4} t^m - k(k-1)(k-2)(k-3)Z_{(k-4)(l+4)m} + \right. \\ &\left. -2k(k-1)(l+4)(l+3)Z_{(k-2)(l+2)m} - m(m-1)Z_{k(l+4)(m-2)} \right] \\ Z_{klm}^4 &= \frac{1}{(m+2)(m+1)} \left[x^k y^l t^{m+2} - k(k-1)(k-2)(k-3)Z_{(k-4)l(m+2)} + \right. \\ &\left. -2k(k-1)l(l-1)Z_{(k-2)(l-2)(m+2)} - l(l-1)(l-2)(l-3)Z_{k(l-4)(m+2)} \right] \end{aligned} \quad (2.21)$$

In formulas (2.21), we put zero instead of a polynomial when any subscript is negative.

3. The method of solving functions

The method of solving functions discussed below belongs to the class of Trefftz methods. As a solution to equation (2.4) we take a linear combination of plate

polynomials which also satisfy this equation. Let us denote these polynomials as

$$\begin{array}{lll} V_1 = P_{00} = 1 & V_2 = P_{10} = x & V_3 = P_{11} = y \\ V_4 = Q_{00} = t & V_5 = P_{20} = \frac{x^2}{2} & V_6 = P_{21} = xy \\ V_7 = P_{22} = \frac{y^2}{2} & V_8 = Q_{10} = xt & \dots \end{array}$$

Obviously, we have one polynomial of order zero, three polynomials of order one, five polynomials of order two and so on. An approximation for the solution to equation (2.4) is

$$u \approx w = \sum_{n=1}^N c_n V_n \quad (3.1)$$

A linear combination (3.1) of plate polynomials satisfies equation (2.4). The coefficients c_n in (3.1) are chosen such that the error (usually in the mean-square sense) for fulfilling given boundary and initial conditions corresponding to equation (2.4) is minimized (see examples). In the case of non-homogeneous equation (2.20), the particular solution has to be added into formula (3.1).

4. Convergence of approximation

Any time we talk about the approximation method we have to ask about the error of approximation and convergence of the procedure. It is easy to specify the error when in formula (3.1) all plate polynomials from the 0-th to K -th order are taken, *e.g.*, for $K = 0$, $N = 1$, for $K = 1$, $N = 1 + 3 = 4$, for $K = 2$, $N = 1 + 3 + 5 = 9$ and so on. Let us consider the plate polynomials obtained in Section 2.4 by using formula (2.19). Notice that when in approximation (3.1) all plate polynomials from the 0-th to $2K$ -th order are taken, then in this approximation:

- full differentials of the order from 0 to K are taken into account,
- some components of differentials of the order from $K + 1$ to $2K$ are taken into account,
- no components of differentials of order larger than $2K$ are taken into account.

Let us denote by w_K approximation (3.1) when all plate polynomials from the 0-th to K -th order are taken. Taking all above into consideration, we can formulate:

Theorem 2. *Let function u satisfy equation (2.4). If in formula (1.1)*

$$\lim_{N \rightarrow \infty} R_N = 0 \quad (4.1)$$

for function u satisfying equation (2.4), with given initial and boundary condition, then

$$\lim_{K \rightarrow \infty} w_K = u$$

Proof. First, we prove that

$$\lim_{K \rightarrow \infty} w_{2K} = u$$

For w_{2K} we have $|R_{2K+1}| \leq |w_{2K} - u| \leq |R_{K+1}|$. But

$$\lim_{K \rightarrow \infty} R_{2K+1} = \lim_{K \rightarrow \infty} R_{K+1} = 0$$

hence

$$\lim_{K \rightarrow \infty} |w_{2K} - u| = 0$$

It means that

$$\lim_{K \rightarrow \infty} w_{2K} = u$$

On the other hand we have $|w_{2K}| \leq |w_{2K+1}| \leq |w_{2(K+1)}|$ hence

$$\lim_{K \rightarrow \infty} w_K = u$$

Condition (4.1) is satisfied for example when all derivatives are commonly restricted. Moreover, the approximation of the solution is better in a small neighbourhood of the considered point. It means that when we expand the function into Taylor series in point $(0, 0, 0)$, the approximation is faster convergent (depending on the number of polynomials) in a small distance from this point. Therefore, we usually look for an approximation if x, y, t are from $(0, 1)$. Theoretically, we get the convergence for any time only if $\lim_{N \rightarrow \infty} R_N = 0$. In practice, for a large value of time we should take into account plenty of polynomials. Therefore, it is recommended to carry out calculations in a short time interval ($\Delta t < 1$). Such a procedure is justified because the solving polynomials contain powers of time and we get a faster convergence if $t < 1$.

5. Plate polynomials in polar coordinates

In the polar coordinates

$$x = r \cos \phi \quad y = r \sin \phi \quad (5.1)$$

equation (2.4) has form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}\right) \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2}\right) + \frac{\partial^2 u}{\partial t^2} = 0 \quad (5.2)$$

Then, to find the plate polynomials in polar coordinates, it is sufficient to substitute in (5.1) polynomials expressed in the Cartesian coordinate system. The polynomials obtained in that way satisfy equation (5.2). We can use recurrent formulas (2.8)-(2.12) in polar coordinates as well, keeping in mind that $x = r \cos \phi$ and $y = r \sin \phi$. The advantage of this method is that in the polar coordinate system we avoid Bessel functions in the solution, which sometimes causes numerical problems.

6. Test example

We will show how the method of solving functions works by discussing a test example. The problem of plate vibration will be solved in two ways. The first is solving the problem in the whole domain. In the second case, the Nodeless Finite Element Method (substructuring) with Trefftz base function will be used. In general, the solving functions can be used as base functions in several variants of FEM. These variants are wider described in the papers by Ciałkowski *et al.* (2007), Maciąg (2007a). A lot of numerical calculations for several equations (Laplace, heat conduction, wave equation) show that solving polynomials as FEM base functions give the best results for Nodeless Finite Element Method (substructuring). Therefore, this approach is proposed here.

Consider vibrations of a square plate described by equation (2.4) and conditions:

— initial conditions

$$u(x, y, 0) = u_0(x, y) = \frac{\sin(\pi x) \sin(\pi y)}{1000} \quad \frac{\partial u(x, y, 0)}{\partial t} = 0 \quad (6.1)$$

— boundary conditions

$$\begin{aligned} u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0 \\ \frac{\partial^2 u(0, y, t)}{\partial x^2} = \frac{\partial^2 u(1, y, t)}{\partial x^2} = \frac{\partial^2 u(x, 0, t)}{\partial y^2} = \frac{\partial^2 u(x, 1, t)}{\partial y^2} = 0 \end{aligned} \quad (6.2)$$

Conditions (6.1) and (6.2) describe a simply supported plate. The exact solution to this problem is

$$u(x, y, t) = \frac{\sin(\pi x) \sin(\pi y) \cos(2\pi^2 t)}{1000}$$

6.1. Approximate solution in the whole domain

The approximated solution $u(x, y, t)$ has the form as follows

$$u \approx w = \sum_{n=1}^N c_n V_n$$

Because the function w satisfies the plate vibration equation (with w being a linear combination of the solving polynomials), in order to find the coefficients c_n , we minimize the norm $\|w - u\|$ for the boundary and initial conditions. We look for an approximate solution u in the time interval $(0, \Delta t)$. Hence, the coefficients c_n have to be chosen appropriately to minimize the functional

$$\begin{aligned} I = & \int_0^1 \int_0^1 \underbrace{\left([w(x, y, 0) - u_0(x, y)]^2 + \left[\frac{\partial w(x, y, 0)}{\partial t} \right]^2 \right)}_{\text{cond. (6.1)}} dx dy + \\ & + \int_0^{\Delta t} \int_0^1 \underbrace{\left([w(0, y, t)]^2 + \left[\frac{\partial^2 w(0, y, t)}{\partial x^2} \right]^2 + [w(1, y, t)]^2 + \left[\frac{\partial^2 w(1, y, t)}{\partial x^2} \right]^2 \right)}_{\text{cond. (6.2)}} dy dt + \\ & + \int_0^{\Delta t} \int_0^1 \underbrace{\left([w(x, 0, t)]^2 + \left[\frac{\partial^2 w(x, 0, t)}{\partial x^2} \right]^2 + [w(x, 1, t)]^2 + \left[\frac{\partial^2 w(x, 1, t)}{\partial x^2} \right]^2 \right)}_{\text{cond. (6.2)}} dx dt \end{aligned}$$

The necessary condition to minimize the functional I is

$$\frac{\partial I}{\partial c_1} = \dots = \frac{\partial I}{\partial c_N} = 0 \quad (6.3)$$

The linear system of equations (6.3) can be written as

$$\mathbf{AC} = \mathbf{B} \quad (6.4)$$

where $\mathbf{C} = [c_1, \dots, c_N]^T$. Note that the matrix \mathbf{A} is symmetric, which simplifies calculations. From equation (6.4), we obtain the coefficients c_n : $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$. In the time intervals $(\Delta t, 2\Delta t)$, $(2\Delta t, 3\Delta t)$, ..., we proceed analogously. Here, the initial condition for the time interval $((m-1)\Delta t, m\Delta t)$ is the value of function w at the end of interval $((m-2)\Delta t, (m-1)\Delta t)$.

All the results shown in Figures 1-4 have been obtained for $(\Delta t = 0.25)$. Figure 1 shows the initial condition (a) the exact one, (b) an approximation

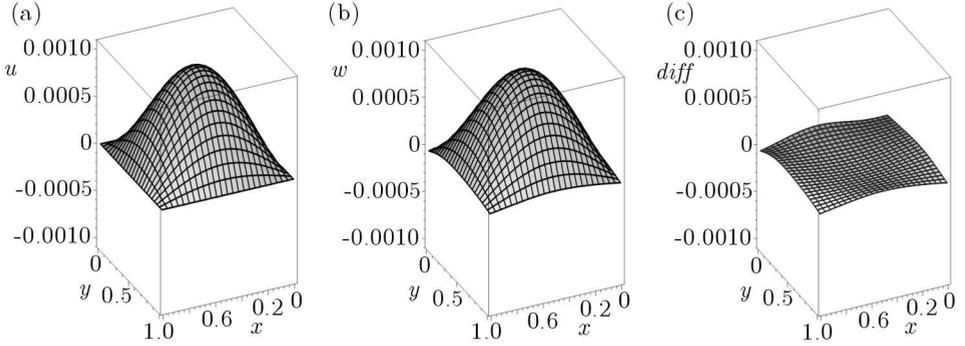


Fig. 1. Solution for time $t = 0$: (a) exact, (b) approximation, (c) difference

with the solving polynomials of degree up to 13, (c) the difference between (a) and (b). It is clear that the approximation is satisfactory.

Figure 2 shows for time $t = 0.15$: (a) the exact solution, (b) an approximation by polynomials of the order 0 up to 13, (c) the difference between (a) and (b).

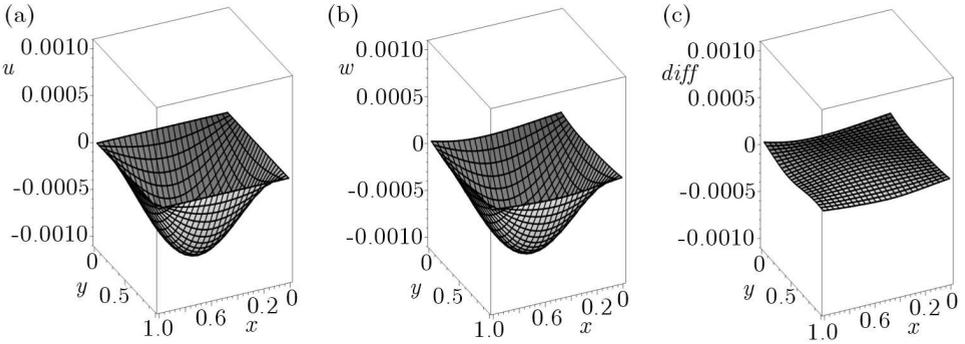


Fig. 2. Solution for time $t = 0.15$: (a) exact, (b) approximation, (c) difference

It is obvious that the approximation presented in the Figs. 1 and 2 is accurate. Figure 3 shows the exact solution (solid line) and the approximation by polynomials of the order from 0 to (a) 5, (b) 10, (c) 15 (dashed line) for the vibration as a function of time at the location $x = 0.5, y = 0.5$.

Figure 3 suggests that the method is convergent. In this case, we can calculate the error of approximation in the norm $L^2(\Omega)$, where $\Omega = (0, \Delta t)$ for the point $x = y = 0.5$. The error is defined as

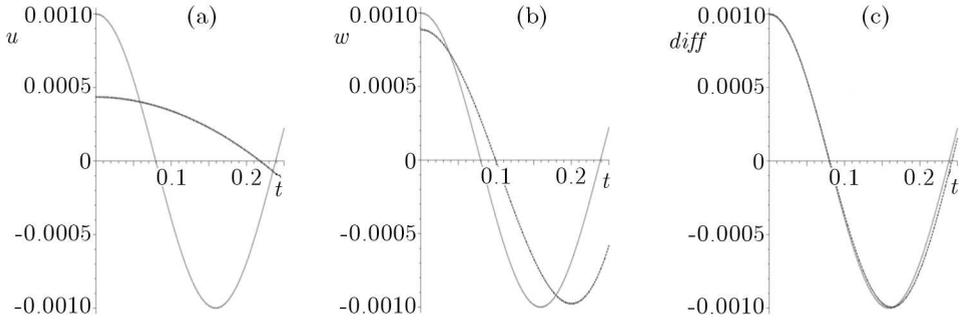


Fig. 3. Exact and approximate solution at the location $x = 0.5$, $y = 0.5$. Approximation by polynomials of the order from 0 to (a) 5, (b) 10, (c) 15

$$E_D = \sqrt{\frac{\int_0^{\Delta t} [u(0.5; 0.5; t) - w_D(0.5; 0.5; t)]^2 dt}{\int_0^{\Delta t} [u(0.5; 0.5; t)]^2 dt}} 100\%$$

In the approximation w_D , we take all plate polynomials from degree 0 to D . Figure 4 shows the error depending on the degree D .

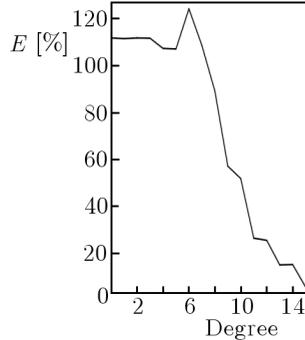


Fig. 4. Error depending on the highest degree D of the polynomial

The error decreases when the number of polynomials in the approximation u increases. This confirms that the method is convergent.

6.2. Nodeless Finite Element Method

The second way of using solving polynomials is the nodeless FEM. Let us divide the domain $(0, 1) \times (0, 1)$ into subdomains Ω_k and seek the solution in

the time-space subdomains $L_k = \Omega_k \times \Delta t$, $k = 1, \dots, K$. Let us introduce a local co-ordinate system in each subdomain L_k and assume the approximation of the solution in the form

$$u_k \approx w_k = \sum_{n=1}^N c_n^k V_n \tag{6.5}$$

To find coefficients c_n^k , we minimize the norm $\|w - u\|$ for the boundary and initial conditions. Moreover, the norms on common borders between neighbouring elements Ω_A and Ω_B : $\|w_A - w_B\|$, $\|\partial w_A / \partial n - \partial w_B / \partial n\|$, $\|\partial^2 w_A / \partial n^2 - \partial^2 w_B / \partial n^2\|$ and $\|\partial^3 w_A / \partial n^3 - \partial^3 w_B / \partial n^3\|$ are minimized simultaneously. Here n is normal to the corresponding border. In the time intervals $(\Delta t, 2\Delta t)$, $(2\Delta t, 3\Delta t), \dots$, we proceed analogously. Here, the value of function w at the end of interval $((m - 2)\Delta t, (m - 1)\Delta t)$ stands for the initial condition for the time interval $((m - 1)\Delta t, m\Delta t)$.

In the nodeless FEM, we can use big elements (subdomains) with a suitable number of base functions. Even for four subdomains, the results are satisfactory. The division of the domain reduces the number of polynomials in an approximation. In order to compare the solution in the whole domain (Section 6.1) with the approximate solution obtained with the nodeless FEM, we assume $\Delta t = 0.125$. The domain is divided into four identical squares. Figure 5 shows the exact solution (solid line) and the approximation (dashed line) for the vibration as a function of time at the location $x = 0.5, y = 0.5$ and the approximation in each subdomain by polynomials of the order from 0 to (a) 5, (b) 9, (c) 10.

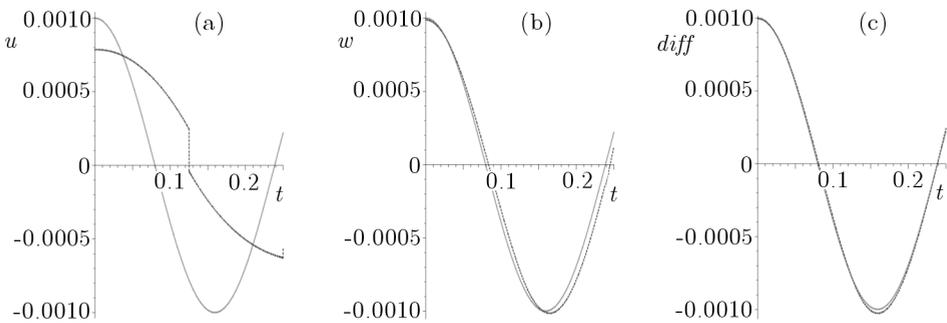


Fig. 5. Exact and approximate solution at the location $x = 0.5, y = 0.5$. Approximation by polynomials of the order from 0 to (a) 5, (b) 9, (c) 10

When comparing Figs. 3 and 5 we notice that for the nodeless FEM the approximation is better than for the solution in the whole domain. Moreover,

using only four subdomains, allowed us to reduce the highest degree of the solving polynomials from 15 down to 10. When using the nodeless FEM, the advantage is such that the elements can be relatively big. Moreover, in each element we have a solution satisfying the equation.

In the considered case we had two time steps. Despite that, the approximation stays good for more steps as well. Figure 6 shows: (a) the exact solution (solid line) and approximation (dashed line) for the vibration as a function of time at the location $x = 0.5, y = 0.5$ and the approximation in each sub-domain by polynomials of the order from 0 to 10, (b) difference between the exact and approximate solution for five time steps $\Delta t = 0.125$.

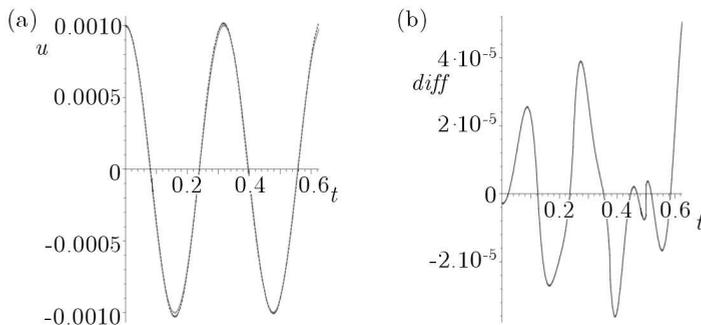


Fig. 6. (a) Exact solution (solid line) and approximation by polynomials of the order from 0 to 10 (dashed line) for $x = 0.5, y = 0.5$, (b) difference between the exact and approximate solution

Like in Section 6.1 we define the relative error of the approximate solution as

$$E_D = \sqrt{\frac{\int_0^{TS\Delta t} [u(0.5; 0.5; t) - w_D(0.5; 0.5; t)]^2 dt}{\int_0^{TS\Delta t} [u(0.5; 0.5; t)]^2 dt}} 100\%$$

where TS is the number of time steps. In the approximation w_D , we take all plate polynomials from degree 0 to D . Table 1 shows values of the error E_{10} in the time interval $(0, TS\Delta t)$ for $\Delta t = 0.125$ depending on the number of time steps TS .

Table 1. Error dependence on the number of time steps

TS	1	2	3	4	5
E_{10} [%]	2.473	2.504	2.718	2.728	2.784

It is clear that the error increases when the number of time steps increases, but still remains small. For 5 time steps, the error does not exceed 3%, which shows that the approximation is good.

7. Concluding remarks

A new simple technique for solving the problems of plate vibration is developed. The main result is the derivation of formulas for plate polynomials (satisfying the plate vibration equation) and their derivatives. Moreover, the convergence of the method is proved. The method of solving functions presented in this paper is a straightforward method for solving plate vibration equations in finite bodies. The author's experience in terms of Trefftz functions for other equations suggests that this method can be also useful when the shape of the plate is more complicated. Then it may be necessary to divide the domain into elements.

The test example shows that the obtained approximation is very good. The solving polynomials presented here can be used in the whole domain or as a base function in the nodeless FEM.

The method of solving functions has three very important properties. The first is that the approximate solution of the considered problem satisfies the governing equation. Secondly, in the nodeless FEM, the elements can be relatively big. Thirdly, the method is relatively flexible in terms of given initial and boundary conditions. These three cases make this method suitable for inverse problems. However, this is a very broad subject and it will be discussed in another paper.

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Funkcje Trefftza dla problemu drgań płyty

Streszczenie

Artykuł przedstawia nową metodę rozwiązywania problemów drgań płyty, które opisywane są cząstkowym równaniem różniczkowym czwartego rzędu. Idea metody polega na uzyskaniu wielomianów spełniających w sposób ścisły dane równanie różniczkowe (funkcje Trefftza). Za rozwiązanie przyjmuje się kombinację liniową tych funkcji. Współczynniki kombinacji liniowej dobierane są w taki sposób, aby zminimalizować błąd dopasowania rozwiązania do warunków początkowo - brzegowych. Podejście proponowane w pracy ma kilka zalet. Po pierwsze liniowa kombinacja funkcji Trefftza spełnia w sposób ścisły dane równanie różniczkowe. Po drugie metoda jest elastyczna odnośnie warunków początkowo - brzegowych. Mogą one być ciągłe lub dyskretne. Dodatkowo warunki te mogą być niekompletne, co czyni metodę użyteczną w rozwiązywaniu zagadnień odwrotnych. Trzecią zaletą metody jest możliwość zastosowania uzyskanych wielomianów jako funkcji bazowych w różnych wariantach Metody Elementów Skończonych. W pracy uzyskano wzory rekurencyjne na funkcje Trefftza oraz ich pochodne dla równania drgań płyty. Zbadano zbieżność metody oraz zamieszczono przykłady numeryczne

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