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NUMERICAL SOLUTIONS OF UNSTEADY BOUNDARY LAYER EQUATIONS FOR A GENERALIZED SECOND GRADE FLUID

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Unsteady, incompressible boundary layer equations for a modified power-law fluid of the second grade are considered. The model is a combination of the power-law and second grade fluid in which the fluid may exhibit normal stresses, shear thinning or shear thickening behaviour. The equations of motion are formulated for two-dimensional flows, and from which the boundary layer equations are derived. By using the similarity transformation, we reduce the boundary layer equations to system of non-linear ordinary differential equation. The ordinary differential equations are numerically integrated for classical boundary layer conditions. Effects of the power-law index and second grade coefficient on the boundary layers are shown.

Key words: power-law fluid of second grade, boundary layers, similarity transformation

1. Introduction

Prandtl's boundary layer theory proved to be of great use in Newtonian fluids as the Navier-Stokes equations can be converted into much simplified equations which are easier to handle. In the past three decades, with the increase of technological importance of non-Newtonian fluids, similar attempts were made at solving the extremely complex equations of motion of non-Newtonian fluids. For these fluids, a boundary layer similar to that of the Newtonian case was assumed *a priori*, and simplified calculations were obtained upon this assumption. It is only recently that well-established mathematical proofs of boundary layers and restrictions on obtaining such boundary layers were outlined.

Non-Newtonian fluids have become more and more industrially important. Polymer solutions, polymer melts, blood, paints and certain oils are the most common examples of non-Newtonian fluids. Several models have been proposed to explain the non-Newtonian behaviour of fluids. Among these, the power-law, differential-type and rate-type models gained much acceptance. Boundary layer assumptions were successfully applied to these models and much work has been done on them.

Power-law fluids are by far the most widely used models to exhibit non-Newtonian behaviour in fluids, and to predict shear thinning and shear thickening behaviour. However, the models have an inadequacy in expressing normal stress behaviour as observed in die swelling and rod climbing in some non-Newtonian fluids. On the other hand, normal stress effects can be expressed in the second grade fluid model, a special type of Rivlin-Ericksen fluids, but this model is incapable of representing shear thinning/thickening behaviour. A fluid model which exhibits all behaviour scenarios is deserved, and Man and Sun (1987) and Man (1992) proposed two models which they called "the power-law fluid of grade 2" and "modified second order (grade) fluid". These models were slight modifications of the usual second grade fluid.

The following power-law fluid of second grade model is considered in the present work

$$\mathbf{T} = -p\mathbf{I} + \Pi^{m/2}(\mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2) \tag{1.1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, p is the pressure, $\mathbf{I}$ is the identity matrix, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin-Ericksen tensors, respectively, μ, m, α_1 and α_2 are material moduli that may be constant or dependent on temperature. For this model, when $m = 0$ and $\alpha_1 = \alpha_2 = 0$, the fluid is Newtonian and hence μ represents the usual viscosity. $m = 0$ corresponds to the standard second grade fluid, $\alpha_1 = \alpha_2 = 0$ corresponds to the power-law fluid. Besides, when $m < 0$, the fluid is shear-thinning, while if $m > 0$, the fluid is shear-thickening. The tensors are defined as

$$\begin{aligned} &\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^\top \qquad\qquad \mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^\top\mathbf{A}_1 \\ &\Pi = \frac{1}{2}\,\mathrm{tr}\,(\mathbf{A}_1^2) \end{aligned} \tag{1.2}$$

where $\mathbf{L} = \mathrm{grad}\,\boldsymbol{v}$, $\boldsymbol{v}$ is the velocity vector and Π is the second invariant of $\mathbf{A}_1$.

Equation (1.1) is in fact a generalization of ordinary second grade fluids with the following constitutive relation

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \tag{1.3}$$

Extensive work has been done on the second grade fluids. A general review of the work is beyond the scope of this study. For the recent works on the model, see Asghar *et al.* (2007), Hayat *et al.* (2007a-e), Hayat and Sajid (2007) for example.

The work on boundary layers of power-law fluids was due to Acrivos *et al.* (1960) and Schowalter (1960). Acrivos *et al.* (1960) examined in detail the flow past a horizontal flat plate including heat transfer. Schowalter (1960) independently developed the two and three-dimensional boundary layer equations and presented some similarity solutions for the equations. Later, the boundary layer treatment due to Pohlhausen was extended to power-law fluids by Bizzell and Slattery (1962). The similarity solution of unsteady boundary layer flow of power-law fluids on a stretching sheet was presented by Yurusoy (2006).

Modifications of the second grade fluids to account for shear thinning/thickening effects were considered in the literature. Man and Sun (1987) first proposed the modifications. Later, Man (1992) considered the unsteady channel flow of a modified second grade fluid, and existence, uniqueness and asymptotic stability of the solutions were exploited. Franchi and Straughan (1993) presented stability analysis of the modified model for a special viscosity function which linearly depended on the temperature. Gupta and Massoudi (1993) investigated the flow of such a fluid with temperature-dependent viscosity between heated plates. Massoudi and Phuoc (2001) studied the flow down a heated inclined plane. The same authors (Massoudi and Phuoc, 2001) analysed a pipe flow by Reynold' s temperature-dependent viscosity model. Hayat and Khan (2005) studied the flow over a porous flat plate and found solutions using the homotopy analysis method (HAM). Recently, a symmetry analysis was presented for the boundary layer equations of the modified second grade fluid (Aksoy*et al.*, 2007). Detailed thermodynamic and stability analyses exist for the second grade (Dunn and Fosdick, 1974) and third grade (Rajagopal and Fosdick, 1980) fluids. Dunn and Rajagopal (1995) presented a critical review of thermodynamic analysis for fluids of differential type, including the models considered here. Many issues regarding the applicability of such non-Newtonian models to real fluids, thermodynamic restrictions imposed on the constitutive equations and doubts raised in the previous literature on these models were addressed in detail.

In this paper, unsteady boundary layer equations for model (1.1) are considered. By using the similarity transformation, the partial differential system is transformed into an ordinary differential system, and the classical boundary conditions are imposed on the equations. These conditions remain invariant under the similarity transformation. Numerical solutions of the ordinary differential equations are found by using finite difference techniques, and the effects of power-law index as well as second grade coefficient on the solutions are outlined.

2. Equations of motion

The mass conservation and linear momentum equations are

$$\operatorname{div} \boldsymbol{v} = 0 \qquad\qquad \rho\frac{d\boldsymbol{v}}{dt} = \operatorname{div} \mathbf{T} + \rho\boldsymbol{b} \tag{2.1}$$

By inserting equations (1.1) and (1.2) into equation (2.2), neglecting body forces, one has

$$\begin{aligned}
&\rho\Big(\boldsymbol{v}_t + \frac{1}{2}\operatorname{grad}|\boldsymbol{v}|^2 + \boldsymbol{\omega}\times\boldsymbol{v}\Big) = -\operatorname{grad} p + \operatorname{grad}\Big[\Big(\frac{1}{2}|\mathbf{A}_1|^2\Big)^{m/2}\Big]\cdot\\
&\cdot(\mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2) + \Big(\frac{1}{2}|\mathbf{A}_1^2\Big)^{m/2}\Big(\mu\Delta\boldsymbol{v} + \alpha_1(\Delta\boldsymbol{\omega}\times\boldsymbol{v}) +\\
&+\alpha_1\Delta\boldsymbol{v}_t + \alpha_1\operatorname{grad}(\boldsymbol{v}\,\Delta\boldsymbol{v}) + \frac{1}{4}(2\alpha_1+\alpha_2)\operatorname{grad}|\mathbf{A}_1|^2 +\\
&+(\alpha_1+\alpha_2)\{\mathbf{A}_1\Delta\boldsymbol{v} + 2\operatorname{div}[(\operatorname{grad}\boldsymbol{v})(\operatorname{grad}\boldsymbol{v})^\top]\}\Big)
\end{aligned} \tag{2.2}$$

where $\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{v}$. For a two dimensional flow in Cartesian coordinates, a straightforward calculation yields the equations of motion as follows

$$\frac{\partial u}{\partial x} + \frac{\partial \boldsymbol{v}}{\partial y} = 0 \tag{2.3}$$

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{m}{2}\Big[4\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)^2\Big]^{\frac{m}{2}-1}\Big\{\Big[8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} +\\
&+2\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x\partial y}\Big)\Big]\Big[2\nu\frac{\partial u}{\partial x} + \frac{\alpha_1}{\rho}\Big(2\frac{\partial^2 u}{\partial x\partial t} + 2u\frac{\partial^2 u}{\partial x^2} + 2v\frac{\partial^2 u}{\partial x\partial y} +\\
&+4\Big(\frac{\partial u}{\partial x}\Big)^2 + 2\frac{\partial v}{\partial x}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)\Big) + \frac{\alpha_2}{\rho}\Big(4\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)^2\Big)\Big] +\\
&+\Big[8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y} + 2\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)\Big(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}\Big)\Big]\Big[\nu\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) +\\
&+\frac{\alpha_1}{\rho}\Big(\frac{\partial^2 u}{\partial y\partial t} + \frac{\partial^2 v}{\partial x\partial t} + u\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x\partial y}\Big) + v\Big(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}\Big) + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} +
\end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&+2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\Big)\Big]\Big\}\Big[4\Big(\frac{\partial u}{\partial x}\Big)^2+\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)^2\Big]^{\frac{m}{2}}\Big\{\nu\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\Big)+\frac{\alpha_1}{\rho}\Big(2\frac{\partial^3 u}{\partial x^2\partial t}+\\
&+\frac{\partial^3 u}{\partial y^2\partial t}+\frac{\partial^3 v}{\partial x\partial y\partial t}-\boldsymbol{v}\frac{\partial^3 v}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+u\frac{\partial^3 u}{\partial x^3}+u\frac{\partial^3 u}{\partial x\partial y^2}+13\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\\
&+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+4\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}+3\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2}+3\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\Big)+\\
&+\frac{\alpha_2}{\rho}\Big(8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}+2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2}+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{m}{2}\Big[4\Big(\frac{\partial u}{\partial x}\Big)^2+\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)^2\Big]^{\frac{m}{2}-1}\Big\{\Big[8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\\
&\quad+2\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}\Big)\Big]\Big[\nu\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)+\frac{\alpha_1}{\rho}\Big(\frac{\partial^2 u}{\partial y\partial t}+\frac{\partial^2 v}{\partial x\partial t}+\\
&\quad+u\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}\Big)+v\Big(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\Big)+2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\Big)\Big]+\\
&\quad+\Big[8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+2\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)\Big(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\Big)\Big]\Big[2\nu\frac{\partial v}{\partial y}+\frac{\alpha_1}{\rho}\Big(2\frac{\partial^2 v}{\partial y\partial t}+\\
&\quad+2u\frac{\partial^2 v}{\partial x\partial y}+2v\frac{\partial^2 v}{\partial y^2}+2\frac{\partial u}{\partial y}\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)+4\Big(\frac{\partial v}{\partial y}\Big)^2\Big)+\\
&\quad+\frac{\alpha_2}{\rho}\Big(4\Big(\frac{\partial v}{\partial y}\Big)^2+\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)^2\Big)\Big]\Big\}+\Big[4\Big(\frac{\partial u}{\partial x}\Big)^2+\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)^2\Big]^{\frac{m}{2}}\cdot\\
&\quad\cdot\Big\{\nu\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\Big)+\frac{\alpha_1}{\rho}\Big(2\frac{\partial^3 v}{\partial y^2\partial t}+\frac{\partial^3 v}{\partial x^2\partial t}+\frac{\partial^3 u}{\partial x\partial y\partial t}u\frac{\partial^3 v}{\partial x^3}+u\frac{\partial^3 v}{\partial x\partial y^2}+\\
&\quad+v\frac{\partial^3 v}{\partial x^2\partial y}+v\frac{\partial^3 v}{\partial y^3}+4\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2}+13\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}+\\
&\quad+3\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial y}\Big)+\frac{\alpha_2}{\rho}\Big(8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}+\\
&\quad+2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial y}+2\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y}\Big)\Big\}
\end{aligned}
\tag{2.5}
$$

where $\nu=\mu/\rho$ is the kinematic viscosity, u and v are x, y components of the velocity inside the boundary layer, t is time, x is the coordinate along the surface and y is the coordinate vertical to the surface. For $m=0$, the equations are reduced to those of the second grade fluid, and for $\alpha_1=\alpha_2=0$, the equations are reduced to those of the power-law fluid. Boundary layer equations will be derived from these equations in the next Section.

3. Boundary layer equations

For the second and third grade fluids, boundary layer equations were derived in detail by Pakdemirli and Suhubi (1992) and Pakdemirli (1992), respectively. Inside the boundary layer, y coordinate is stretched by a factor $1/\delta$

$$Y = \frac{y}{\delta} \tag{3.1}$$

As in the usual boundary layer assumption, x, u and p are of order 1 and v is of order δ. For a standard boundary layer of one inner deck and one outer deck, the dimensionless coefficients are required to be as follows

$$\nu = \overline{\nu}\delta^{m+2} \qquad \alpha_1 = \varepsilon_1\delta^{m+2} \qquad \alpha_2 = \varepsilon_2\delta^{m+2} \tag{3.2}$$

Other choices are also possible which may lead to multiple-deck boundary layers. Multiple-deck boundary layer analysis is beyond the scope of this work. For the second and third grade fluids, such analysis has been presented elsewhere (Pakdemirli, 1994).

With the above assumptions and keeping the terms of order 1 in continuity and x-momentum and order $1/\delta$ in the y-momentum, one finally has

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial Y} = 0 \tag{3.3}$$

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{m}{2}\Big[\Big(\frac{\partial u}{\partial Y}\Big)^2\Big]^{\frac{m}{2}-1}\Big\{2\varepsilon_2\Big(\frac{\partial u}{\partial Y}\Big)^3\frac{\partial^2 u}{\partial x\partial Y} + \\
&\quad +2\frac{\partial u}{\partial Y}\frac{\partial^2 u}{\partial Y^2}\Big[\overline{\nu}\frac{\partial u}{\partial Y} + \varepsilon_1\Big(\frac{\partial^2 u}{\partial Y\partial t} + u\frac{\partial^2 u}{\partial x\partial Y} + v\frac{\partial^2 u}{\partial Y^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial Y}\Big)\Big]\Big\} + \\
&\quad +\Big(\frac{\partial u}{\partial Y}\Big)^m\Big\{\overline{\nu}\frac{\partial^2 u}{\partial Y^2} + \varepsilon_1\Big(\frac{\partial^3 u}{\partial Y^2\partial t} + v\frac{\partial^3 u}{\partial Y^3} + u\frac{\partial^3 u}{\partial x\partial Y^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial Y^2} + \\
&\quad +3\frac{\partial u}{\partial Y}\frac{\partial^2 u}{\partial x\partial Y}\Big) + 2\varepsilon_2\frac{\partial u}{\partial Y}\frac{\partial^2 u}{\partial x\partial Y}\Big\}
\end{aligned} \tag{3.4}$$

$$-\frac{\partial p}{\partial Y} + (m+2)(2\varepsilon_1 + \varepsilon_2)\Big(\frac{\partial u}{\partial Y}\Big)^{m+1}\frac{\partial^2 u}{\partial Y^2} = 0 \tag{3.5}$$

If the new pressure is defined as

$$\overline{p} = p - (2\varepsilon_1 + \varepsilon_2)\Big(\frac{\partial u}{\partial Y}\Big)^{m+2} \tag{3.6}$$

equation (3.5) reduces to $\partial\overline{p}/\partial Y = 0$, and hence $\overline{p} = \overline{p}(x,t)$. But matching with the inviscid outer solution requires

$$\frac{\partial \overline{p}}{\partial x} = -\frac{\partial U}{\partial t} - U\frac{\partial U}{\partial x} \tag{3.7}$$

Differentiation of equation (3.6) with respect to x using equation (3.7), and substitution into equation (3.4) finally yields the boundary layer equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial Y}=0 \tag{3.8}$$

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial Y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\overline{\nu}(m+1)\Big(\frac{\partial u}{\partial Y}\Big)^m\frac{\partial^2 u}{\partial Y^2}+\frac{\varepsilon_1}{\rho}\Big(\frac{\partial u}{\partial Y}\Big)^{m-1}\cdot\\ \cdot\Big\{m\Big[\frac{\partial^2 u}{\partial Y^2}\Big(\frac{\partial^2 u}{\partial Y\partial t}+u\frac{\partial^2 u}{\partial x\partial Y}+v\frac{\partial^2 u}{\partial Y^2}+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial Y}\Big)-2\Big(\frac{\partial u}{\partial Y}\Big)^2\frac{\partial^2 u}{\partial x\partial Y}\Big]+\\ +\frac{\partial u}{\partial Y}\Big(\frac{\partial^3 u}{\partial Y^2\partial t}+v\frac{\partial^3 u}{\partial Y^3}+u\frac{\partial^3 u}{\partial x\partial Y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial Y^2}-\frac{\partial u}{\partial Y}\frac{\partial^2 u}{\partial x\partial Y}\Big)\Big\}\end{aligned} \tag{3.9}$$

The boundary conditions for the problem are

$$\begin{aligned} &u(x,0,t)=0 \qquad && v(x,0,t)=0\\ &u(x,\infty,t)=U(x,t) && \frac{\partial u}{\partial y}(x,\infty,t)=0\end{aligned} \tag{3.10}$$

where $U(x,t)$ is the velocity outside the boundary layer. For $m=0$, the equations represent the boundary layers of the standard second grade fluid, and for $\varepsilon_1=0$ the equations represent the boundary layers of the power-law fluid.

4. Similarity solutions

In this section, the similarity solutions to equations (3.8) and (3.9) will be presented. Equations (3.8) and (3.9) have the similarity transformation

$$\begin{aligned} &\xi=\overline{\nu}^{-\frac{1}{m+2}}t^{\frac{m-1}{m+2}}x^{-\frac{m}{m+2}}Y \qquad && u=\frac{x}{t}f(\xi)\\ &v=\overline{\nu}^{\frac{1}{m+2}}t^{-\frac{2m+1}{m+2}}x^{\frac{m}{m+2}}g(\xi) && U=\lambda\frac{x}{t}\end{aligned} \tag{4.1}$$

where λ is a constant. By substituting equation (4.1) into equations (3.8), (3.9) and (3.10) and rearranging, we finally obtain the corresponding ordinary differential equations

$$f-\frac{m}{m+2}\xi f'+g'=0 \tag{4.2}$$

$$
\begin{aligned}
&-f+\frac{m-1}{m+2}\xi f'+f^2-\frac{m}{m+2}\xi f'+gf'=(m+1)|f'|^{m-1}f'f''+\\
&+k_1|f'|^{m-1}\Big[-2f'f''+\frac{m^2-m}{m+2}\xi f''^2+2(m+1)ff'f''-\frac{m^2}{m+2}\xi ff''^2+\\
&+mgf''^2-2\frac{2m+1}{m+2}f'^3+gf'f'''-\frac{m}{m+2}\xi ff'f'''+\frac{m-1}{m+2}\xi f'f'''\Big]-\lambda+\lambda^2
\end{aligned}
\tag{4.3}
$$

where the primes denote differentiation with respect to the similarity variable ξ, and $k_1 = \varepsilon_1/\rho\overline{\nu}t$ is the dimensionless second grade parameter.

Note that one of the problems in dealing with non-Newtonian boundary layers is the paucity of boundary conditions. The problem arises in the differential and rate type of fluids and is discussed in detail by Rajagopal *et al.* (1986). In this problem, the last boundary condition is added to the set. Under the selected similarity transformations, the appropriate boundary conditions become

$$
\begin{aligned}
f(0)&=0 \qquad\qquad & g(0)&=0\\
f(\infty)&=\lambda \qquad\qquad & f'(\infty)&=0
\end{aligned}
\tag{4.4}
$$

By using a special finite difference scheme, equations (4.2) and (4.3) are integrated subject to boundary conditions (4.4). In Figs. 1a,b, functions f and g related to the x and Y component of the velocities are drawn for various values of k_1. An increase in this coefficient results in a decrease in f but an increase in g. The boundary layer thickens as k increases. In Figs. 2a,b, the shear thickening case $(m > 0)$ is presented. Numerical results for various power-law index m are plotted for $f(0) = \lambda = 1$. In Fig. 2a, the f function and in Fig. 2b the g function is plotted for m values 0, 0.2, 0.4. Both functions which are related to the velocity components are observed to increase with increasing m. It is interesting to note that the velocity profiles are intersected with each other in the near-wall region as highlighted by insertion of Fig. 2a, where these intersections are found to occur at about $\zeta = 1.7$ for the specified parameters. Figure 2a shows that the boundary layer thickness decreases as m increases for $1.7 < \zeta < \infty$. For the shear thinning values (i.e. $m < 0$), the similarity functions related to the x and Y velocity components are given in Figs. 3a,b, respectively. Figure 3a shows that the x component of the velocity decreases as m decreases for $1.8 < \zeta < \infty$. It concludes that the boundary layer becomes thinner with increasing m for $1.8 < \zeta < \infty$. In Fig. 3b, the Y component of the velocity is observed to increase with increasing m.

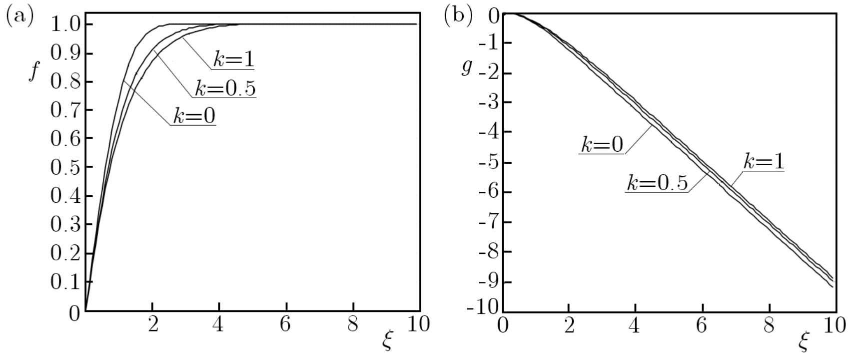

Fig. 1. Influence of the second grade coefficient k_1 on the similarity function related to the x-component velocity (a) and the Y-component velocity (b) ($m = 0.1$)

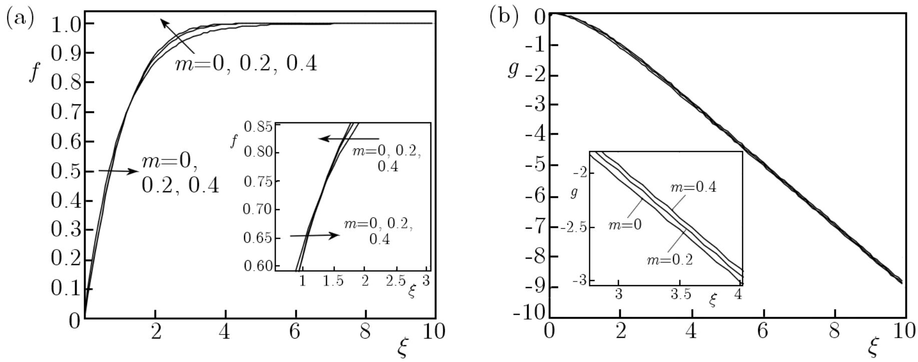

Fig. 2. Influence of the positive power-law index m on the similarity function related to the x-component velocity (a) and the Y-component velocity (b) ($k_1 = 1$)

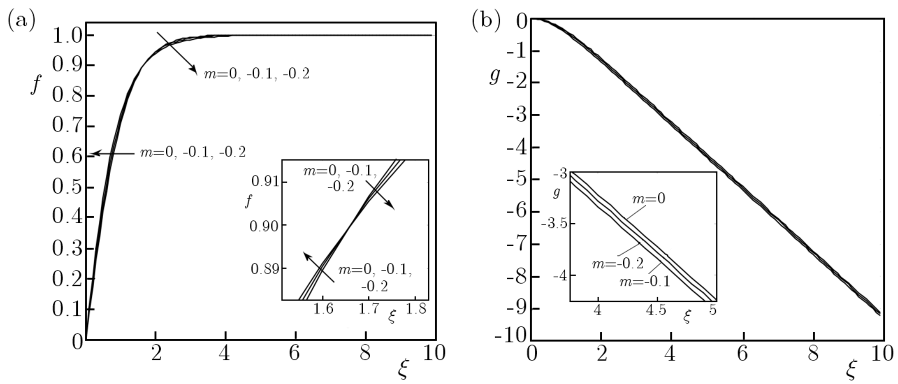

Fig. 3. Influence of the negative power-law index m on the similarity function related to the x-component velocity (a) and the Y-component velocity (b) ($k_1 = 1$)

5. Conclusions

In this study, two-dimensional unsteady boundary layer equations for a power-law fluid of the second grade which can exhibit shear thinning/thickening behaviour for classical boundary conditions are derived. The power-law equations and second grade equations can be retrieved from the given equations as special cases. Using suitable transformations, the governing equations are transformed into nonlinear ordinary differential equations and are solved numerically by a finite difference scheme approximation. The effect of power-law index m and second grade coefficient on the solutions is investigated. An increase in the second grade coefficient leads to a thicker boundary layer. The qualitative behaviour of boundary layers of the modified second grade fluid looks similar to that of the power law fluid for shear thinning and shear thickening cases.

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Numeryczne rozwiązania równań niestacjonarnej warstwy przyściennej uogólnionego płynu drugiego rzędu

Streszczenie

W pracy omówiono równania niestacjonarnej i nieściśliwej warstwy przyściennej zmodyfikowanego modelu płynu drugiego rzędu typu potęgowego. Rozważany model stanowi kombinację koncepcji płynu drugiego rzędu i opisu potęgowego, która pozwala na odzwierciedlenie zjawiska występowania naprężeń normalnych w płynie oraz efektu zmiany grubości warstwy pod wpływem naprężeń stycznych. Sformułowano równania ruchu dla przepływu dwuwymiarowego i na ich podstawie wyprowadzono równania warstwy przyściennej. Używając przekształcenia przez podobieństwo, uproszczono równania warstwy do układu nieliniowych równań różniczkowych zwyczajnych. Następnie równania te scałkowano numerycznie, stosując klasyczne warunki brzegowe. W dalszej części przeanalizowano wpływ wykładnika potęgowego modelu oraz współczynnika drugiego rzędu na zachowanie się płynu w warstwie przyściennej.

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