

## NON-LINEAR ANALYSIS OF DYNAMIC STABILITY OF METAL FOAM CIRCULAR PLATE

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The subject of the paper is a circular plate under radial compression. The plate is made of metal foam. Properties of the plate vary across its thickness. The middle plane of the plate is its symmetry plane. The field of displacement of any cross section of the plate, nonlinear components of the strain field and the stress field are defined. Basing on the Hamilton principle, a system of differential equations of dynamic stability of the plate is formulated. This basic system of equations is approximately solved. The results of the studies are compared to the homogeneous circular plate and shown in figures.

*Key words:* metal foam plate, critical load, dynamic

### 1. Introduction

Contemporary structures are manufactured, among other things, of metal foam. Banhart (2001) presented the manufacture, characterisation and application of cellular metals and metal foams for structures. Carrera (2000, 2001, 2003) reviewed the broken-line hypothesis and generalized it to multi-layered constructions. Carrera *et al.* (2008) described the static analysis of functionally graded material plates subjected to transverse mechanical loadings. Wang *et al.* (2000) discussed in details the shear deformable effect of beams and plates and described some special theories of them. Magnucka-Blandzi (2006, 2008, 2009) carried out analytical investigations of strength, stability and free vibrations of porous-cellular plates with consideration of a non-linear hypothesis of deformation of flat cross section of the structures. Volmir (1972) collected and presented problems of non-linear vibrations and dynamic stability of thin plates and shells. Moreover, the dynamic criterion of stability was formulated. Kowal-Michalska (2006) described dynamic stability of plates

including criterions of: Volmir, Budiansky-Hutchinson, Ari-Gur and Weller, Petry and Fahlbush.

In Magnucka-Blandzi (2009), linear analysis of the plate was studied. Instead, this paper is concerned with non-linear analysis of the porous circular plate. The paper is an improvement and continuation of the papers by Magnucka-Blandzi (2006, 2008, 2009). The plate with radius  $R$  and thickness  $h$  carries a radial compressive force  $N(t)$ .

## 2. Physical model of the plate

An isotropic porous circular plate with the clamped edge under radial compression is studied. The plate with a simply supported edge can also be analysed in a similar way. The plate is made of the metal foam. The plate is porous inside and the material is of continuous mechanical properties varying in the normal direction (Fig. 1). A degree of porosity and Young's modulus vary thro-

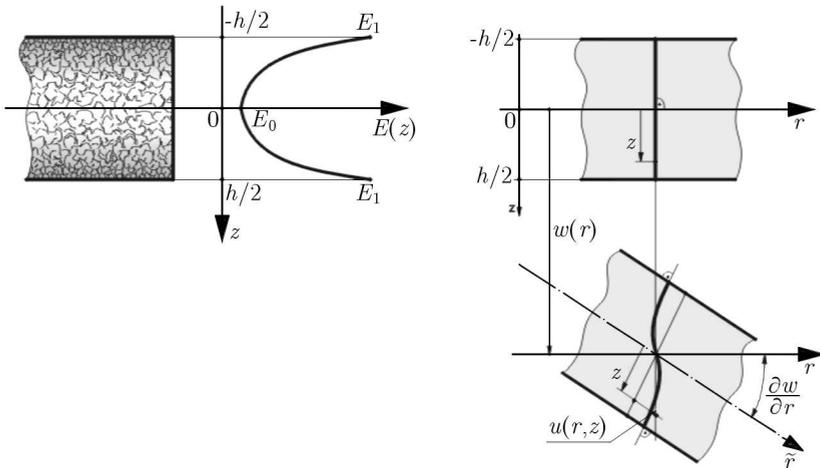


Fig. 1. Schemes of the plate and deformation of its plane cross section

ugh the thickness of the plate. The minimal values are in the middle surface of the plate. The maximal values occur at its top and bottom surfaces. For such a case, the Kirchhoff and Mindlin plate theories do not correctly determine displacements of the plate cross-section. Wang *et al.* (2000) discussed in details the effect of non-dilatational strain of middle layers on bending of plates subject to various load cases. Magnucka-Blandzi and Magnucki (2007),

Magnucka-Blandzi (2006, 2008, 2009) thoroughly described the non-linear hypothesis of deformation of the plate cross section. The moduli of elasticity and mass density are defined as follows

$$\begin{aligned} E(z) &= E_1[1 - e_0 \cos(\pi\zeta)] \\ G(z) &= G_1[1 - e_0 \cos(\pi\zeta)] \\ \varrho(z) &= \varrho_1[1 - e_m \cos(\pi\zeta)] \end{aligned} \quad (2.1)$$

where

- $e_0$  – porosity coefficient of elasticity moduli,  $e_0 = 1 - E_0/E_1$
- $e_m$  – dimensionless parameter of mass density,  $e_m = 1 - \varrho_0/\varrho_1$
- $E_0, E_1$  – Young's modulus at  $z = 0$  and  $z = \pm h/2$ , respectively
- $G_0, G_1$  – shear modulus for  $z = 0$  and  $z = \pm h/2$ , respectively
- $G_j$  – relationship between moduli of elasticity for  $j = 0, 1,$   
 $G_j = E_j/[2(1 + \nu)]$
- $\nu$  – Poisson's ratio (constant for the entire plate)
- $\varrho_0, \varrho_1$  – mass densities for  $z = 0$  and  $z = \pm h/2$ , respectively
- $\zeta$  – dimensionless coordinate,  $\zeta = z/h$
- $h$  – thickness of the plate.

Choi and Lakes (1995) presented mechanical properties for porous materials. Basing on their results, the following relationship is defined:  $e_m = 1 - \sqrt{1 - e_0}$ . Magnucka-Blandzi and Magnucki (2007), Magnucki *et al.* (2006), Magnucki and Stasiewicz (2004a,b) proposed a non-linear hypothesis of the cross-section deformation of the structure wall. Deformation of any plane cross section is shown in Fig. 1. Applying this hypothesis, the displacements are assumed in the same form as in Magnucka-Blandzi (2009)

$$\begin{aligned} u(r, z, t) &= \\ &+ - h \left\{ \zeta \frac{\partial w}{\partial r} - \frac{1}{\pi} [\psi_1(r, t) \sin(\pi\zeta) + \psi_2(r, t) \sin(2\pi\zeta) \cos^2(\pi\zeta)] \right\} \end{aligned} \quad (2.2)$$

where  $\psi_1(r, t)$ ,  $\psi_2(r, t)$  are dimensionless functions of displacements. If  $\psi_1(r, t) = \psi_2(r, t) = 0$ , the field of displacement  $u$  is the linear Kirchhoff-Love hypothesis.

The nonlinear geometric relationships, i.e. components of the strain are

$$\begin{aligned}
 \varepsilon_r &= \frac{\partial u}{\partial r} + \frac{1}{2} \left( \frac{\partial w}{\partial r} \right)^2 = \\
 &= -h \left\{ \zeta \frac{\partial^2 w}{\partial r^2} - \frac{1}{\pi} \left[ \frac{\partial \psi_1}{\partial r} \sin(\pi \zeta) + \frac{\partial \psi_2}{\partial r} \sin(2\pi \zeta) \cos^2(\pi \zeta) \right] \right\} + \frac{1}{2} \left( \frac{\partial w}{\partial r} \right)^2 \\
 \varepsilon_\varphi &= \frac{u}{r} = \\
 &= -h \left\{ \frac{1}{r} \zeta \frac{\partial w}{\partial r} - \frac{1}{\pi} \left[ \frac{1}{r} \psi_1(r, t) \sin(\pi \zeta) + \frac{1}{r} \psi_2(r, t) \sin(2\pi \zeta) \cos^2(\pi \zeta) \right] \right\} \\
 \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \psi_1(r, t) \cos(\pi \zeta) + \psi_2(r, t) [\cos(2\pi \zeta) + \cos(4\pi \zeta)]
 \end{aligned} \tag{2.3}$$

where  $\varepsilon_r$  is the normal strain along the  $r$ -axis,  $\varepsilon_\varphi$  is the circular strain, and  $\gamma_{rz}$  – the shear strain. Basing on Hooke's law, the stresses were defined.

### 3. Mathematical model of the plate

#### 3.1. Potential energy and work of the load

Equations of dynamic stability are formulated basing on Hamilton's principle

$$\delta \int_{t_1}^{t_2} (T - U_\varepsilon + W) dt = 0 \tag{3.1}$$

where  $T$  denotes kinetic energy, which is approximately formulated. It only includes deflections  $w(r, t)$  without tangent displacements  $u(r, t)$ .

$$T = \pi h \int_0^R \int_{-\frac{1}{2}}^{\frac{1}{2}} r \varrho(\zeta) \left( \frac{\partial w}{\partial t} \right)^2 d\zeta dr$$

$U_\varepsilon$  is the energy of elastic strain

$$U_\varepsilon = \pi h \int_0^R \int_{-1/2}^{1/2} r (\sigma_r \varepsilon_r + \sigma_\varphi \varepsilon_\varphi + \tau_{rz} \gamma_{rz}) d\zeta dr$$

$W$  is the work which follows from the compressive force

$$W = \pi N(t) \int_0^R r \left( \frac{\partial w}{\partial r} \right)^2 dr$$

$R$  is the radius of the plate,  $\rho$  – mass density of the plate,  $t_1, t_2$  – initial and final times,  $N(t)$  – intensity of the compressive force. In the numerical calculations, the intensity of compressive force is assumed as follows

$$N(t) = N_0 \sin^2\left(\frac{1}{2}\theta t\right)$$

or

$$N(t) = N_0 \frac{t}{t_0}$$

where  $\theta = \pi/t_0$ ,  $t_0$  – the initial time. These forces have unchanging direction, whereas the first one is an impulsive compressive force and the second one steadily increases.

### 3.2. Equations of stability

Taking into account principle (3.1), the system of three stability equations of motion for the porous plate under compression is formulated in the following form

$$\begin{aligned} (\delta w) \quad & \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( c_0 \frac{\partial w}{\partial r} - c_1 \psi_1 - c_2 \psi_2 \right) \right] \right\} \right\} - \frac{1}{h^2} c_9 \frac{\partial}{\partial r} \left[ r \left( \frac{\partial w}{\partial r} \right)^3 \right] + \\ & + 4 \frac{1 - \nu^2}{E_1 h^3} \left[ \pi N(t) \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) + c_{10} \rho_1 r h \frac{\partial^2 w}{\partial t^2} \right] = 0 \\ (\delta \psi_1) \quad & \frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( c_1 \frac{\partial w}{\partial r} - c_3 \psi_1 - c_4 \psi_2 \right) \right] \right\} + \frac{1 - \nu}{h^2} (c_5 \psi_1 + c_6 \psi_2) = 0 \quad (3.2) \\ (\delta \psi_2) \quad & \frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( c_2 \frac{\partial w}{\partial r} - c_4 \psi_1 - c_7 \psi_2 \right) \right] \right\} + \frac{1 - \nu}{h^2} (c_6 \psi_1 + c_8 \psi_2) = 0 \end{aligned}$$

where

$$\begin{aligned} c_0 &= \frac{\pi^3 - 6e_0(\pi^2 - 8)}{3\pi^2} & c_1 &= \frac{8 - \pi e_0}{\pi^2} \\ c_2 &= \frac{225\pi - 512e_0}{300\pi^2} & c_3 &= 2 \frac{3\pi - 4e_0}{3\pi^2} \\ c_4 &= \frac{64 - 15\pi e_0}{30\pi^2} & c_5 &= \frac{3\pi - 8e_0}{3} \\ c_6 &= \frac{32 - 15\pi e_0}{30} & c_7 &= \frac{1575\pi - 4096e_0}{2520\pi^2} \\ c_8 &= 2 \frac{315\pi - 832e_0}{315} & c_9 &= 2(\pi - 2e_0) \\ c_{10} &= \frac{\pi - 2e_m}{3} \end{aligned}$$

The boundary conditions for the plate with the clamped edge are

$$\begin{aligned} w(R, t) = 0 & & \psi_1(0, t) = \psi_2(0, t) = 0 \\ \frac{\partial w}{\partial r} \Big|_{r=R} = 0 & & \frac{\partial w}{\partial r} \Big|_{r=0} = 0 \end{aligned} \quad (3.3)$$

where  $M_r = \int_{-h/2}^{h/2} z \sigma_r dz$  is the radial bending moment. The system of differential equations (3.2) includes three unknown functions, which are assumed in forms

$$\begin{aligned} \psi_1(r, t) &= -6\psi_{a1} \left[ \left( \frac{r}{R} \right) - \left( \frac{r}{R} \right)^2 \right] \\ \psi_2(r, t) &= -6\psi_{a2} \left[ \left( \frac{r}{R} \right) - \left( \frac{r}{R} \right)^2 \right] \\ w(r, t) &= w_a(t) \left[ 1 - 3 \left( \frac{r}{R} \right)^2 + 2 \left( \frac{r}{R} \right)^3 \right] \end{aligned} \quad (3.4)$$

These functions satisfy boundary conditions (3.3). Substituting them into system (3.2) and using Galerkin's method, one obtains a system of three equations in the form

$$\begin{aligned} \left[ c_0 - \frac{4\pi(1-\nu^2)R^2}{15E_1h^3} N(t) \right] w_a(t) - c_1 R \psi_{a1} - c_2 R \psi_{a2} + \frac{4}{35} c_9 \frac{1}{h^2} w_a^3(t) + \\ + c_{10} \varrho_1 \frac{12(1-\nu^2)R^4}{105E_1h^2} \frac{d^2 w_a}{dt^2} = 0 \\ c_1 w_a - c_{13} R \psi_{a1} - c_{12} R \psi_{a2} = 0 \\ c_2 w_a - c_{12} R \psi_{a1} - c_{11} R \psi_{a2} = 0 \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} c_{11} &= c_7 + c_8 c_{14} & c_{12} &= c_4 + c_6 c_{14} \\ c_{13} &= c_3 + c_5 c_{14} & c_{14} &= \frac{(1-\nu)R^2}{15h^2} \end{aligned}$$

From the second and third equations of system (3.5),  $\psi_{a1}$ ,  $\psi_{a2}$  functions may be calculated, namely

$$\psi_{a1} = \tilde{\psi}_{a1} \frac{w_a}{R} \quad \psi_{a2} = \tilde{\psi}_{a2} \frac{w_a}{R} \quad (3.6)$$

where

$$\tilde{\psi}_{a1} = \frac{c_1 c_{11} - c_2 c_{12}}{c_{13} c_{11} - c_{12}^2} \quad \tilde{\psi}_{a2} = \frac{c_2 c_{13} - c_1 c_{12}}{c_{13} c_{11} - c_{12}^2}$$

Substitution of functions (3.6) into the first equation of system (3.5) yields the second order nonlinear differential equation of motion in the following form

$$\frac{d^2 w_a}{dt^2} + \frac{c_9}{c_{10}(1-\nu^2)} \frac{E_1}{\rho_1 R^4} w_a^3(t) + \frac{7\pi N_{cr}}{3c_{10}\rho_1 h R^2} \left(1 - \frac{N_t}{N_{cr}}\right) w_a(t) = 0 \quad (3.7)$$

where

$$N_{cr} = \frac{15E_1 h^3}{4\pi(1-\nu^2)R^2} (c_0 - c_1 \tilde{\psi}_{a1} - c_2 \tilde{\psi}_{a2})$$

is the intensity of the critical force ( $N_{cr}$  [N/mm]). Galerkin's method allowed one to reduce this problem of a continuous structure, circular plate, to a discrete problem with a single degree of freedom.

In a particular case, the static equilibrium path follows from equation (3.7) in the form

$$N(t) = \frac{15}{4\pi(1-\nu^2)} \left[ c_0 - c_1 \tilde{\psi}_{a1} - c_2 \tilde{\psi}_{a2} + \frac{4}{35} c_9 \left(\frac{w_a}{h}\right)^2 \right] \frac{E_1 h^3}{R^2}$$

#### 4. Numerical calculations

Some examples will be given below for a family of plates with height  $h = 10$  mm, radius  $R = 1500$  mm, Young's modulus  $E_1 = 7100$  MPa and mass density  $\rho_1 = 2.7 \cdot 10^{-7}$  kg/mm<sup>3</sup>. The influence of porosity coefficient of elasticity moduli and the influence of compressive force on the amplitude of displacement is studied. Two kinds of radial compressive forces are assumed. Their plots are shown in Fig. 2. These two loads are in forms

$$N^{(1)}(t) = N_0 \frac{t}{t_0} \quad N^{(2)}(t) = N_0 \sin^2\left(\frac{1}{2}\theta t\right)$$

where  $\theta = \pi/t_0$ .

In the first example, the intensity of compressive force is linear and assumed in form  $N(t) = N_0 t/t_0$ , where  $N_0 = N_{cr}$ ,  $t_0 = 3$  (the initial time). Static and dynamic equilibrium paths are presented in Fig. 3 for the homogeneous plate ( $e_0 = 0$ ) and for the non-homogeneous plate ( $e_0 = 0.8$ ).

In the second example, only the homogeneous plate is considered. The intensity of compressive force is assumed as a pulsating compressive force in form  $N(t) = N_0 \sin^2(\theta t/2)$ , where  $\theta = \pi/t_0$ ,  $t_0 = 3$  and  $N_0 = kN_{cr}$  ( $k = 1.2, 1.5, 1.8$ ). Static and dynamic equilibrium paths are presented in

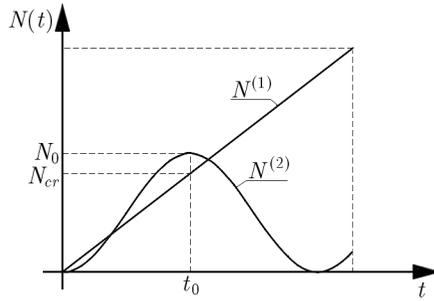


Fig. 2. Radial intensity of compressive forces

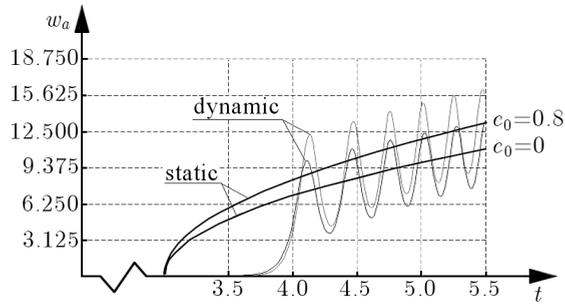


Fig. 3. Amplitudes of deflections for homogeneous and non-homogeneous plates

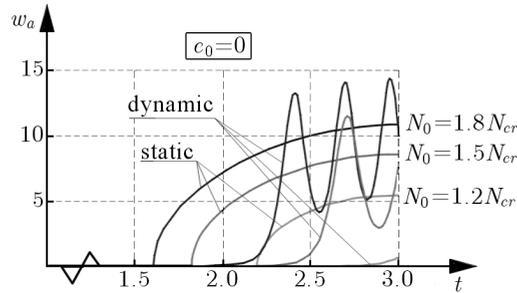


Fig. 4. Amplitudes of deflections for the homogeneous plate

Fig. 4. The influence of the pulsating compressive force on the amplitude of displacement is shown.

In the last example, homogeneous and non-homogeneous plates are compared. In Fig. 5, the plots of equilibrium paths are shown. The pulsating compressive force is the same as previously, but  $N_0 = 1.8N_{cr}$ .

It could be noticed that in the post-buckling state vibrations of the plates around the static equilibrium paths for homogeneous and non-homogeneous

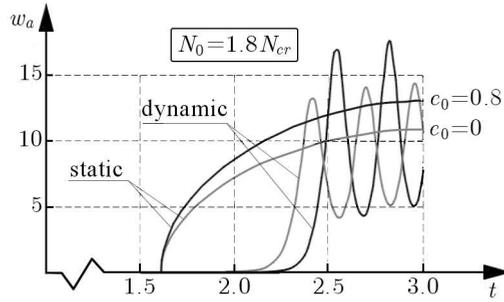


Fig. 5. Amplitudes of deflections for homogeneous and non-homogeneous plates

plates appear as well. The above results of numerical analysis issue from the simplified circular plate model. Despite of this simplification, the behaviour of the plate under dynamic loads could be useful in practice.

## 5. Conclusions

- The metal foam circular plate is a generalization of sandwich or multi-layer plates.
- Correct hypotheses of plane cross sections for homogeneous plates are useless in the case of a porous-cellular plate as elastic constants vary considerably along its depth.
- The non-linear hypothesis of deformation of the flat cross section of the plate is optional to hypothesis presented by Wang *et al.* (2000) or Carrera (2001, 2003) and Carrera *et al.* (2008) and it includes:
  - linear hypothesis for homogeneous plates
  - shear deformable effect.
- The mathematical model of dynamic stability of the metal foam circular plate could be reduced to a single differential equation of motion.
- The dynamic equilibrium path is the solution to differential equation of motion (3.7).
- In a particular case, the static equilibrium path follows from equation of motion (3.7).

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### **Analiza nieliniowa stateczności dynamicznej płyty kołowej wykonanej z piany metalowej**

#### Streszczenie

Przedmiotem pracy jest płyta kołowa obciążona promieniowo. Płyta wykonana jest z piany metalowej. Właściwości mechaniczne płyty są zmienne na jej grubości. Płaszczyzna środkowa płyty jest jej płaszczyzną symetrii. Zdefiniowano pole przemieszczeń dla dowolnego przekroju poprzecznego płyty, nieliniowe odkształcenia oraz naprężenia. Układ równań różniczkowych stateczności dynamicznej płyty zdefiniowano na podstawie zasady Hamiltona. Układ ten rozwiązano w sposób przybliżony. Wyniki badań numerycznych tej płyty porównano z odpowiednimi wielkościami dla płyty jednorodnej i przedstawiono na rysunkach.

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