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STOCHASTIC HOPF BIFURCATION OF QUASI-INTEGRABLE HAMILTONIAN SYSTEMS WITH TIME-DELAYED FEEDBACK CONTROL

Zhong Hua Liu

Xiamen University, Department of Civil Engineering, Xiamen, China
e-mail: zhliuzju@yahoo.com

Wei Qiu Zhu

Zhejiang University, Department of Mechanics, State Key Laboratory of Fluid Power Transmission and Control, Hangzhou, China
e-mail: wqzhu@yahoo.com (corresponding author)

The stochastic Hopf bifurcation of multi-degree-of-freedom (MDOF) quasi-integrable Hamiltonian systems with time-delayed feedback control subject to Gaussian white noise excitations is studied. First, the time-delayed feedback control forces are approximately expressed in terms of the system state variables without time delay, and the system is converted into anordinary quasi-integrable Hamiltonian system. The averaged Itô stochastic differential equations are derived by using the stochastic averaging method for quasi-integrable Hamiltonian systems. Then, an expression for the average bifurcation parameter of the averaged system is obtained approximately, and a criterion for determining the stochastic Hopf bifurcation caused by the time-delayed feedback control forces in the original system as the value of the average bifurcation parameter changing is proposed. An example is worked out in detail to illustrate the above criterion and its validity, and to show the effect of the time delay in the feedback control on the stochastic Hopf bifurcation of the system.

Key words: stochastic Hopf bifurcation, quasi-integrable Hamiltonian system, stochastic averaging, time-delayed feedback control

1. Introduction

Time delay is usually unavoidable in feedback control systems due to the time spent for measuring and estimating of the system state, calculating and executing of the control forces, etc. This time delay often leads to instability or poor performance of controlled systems. Thus, the issue of handling the time delay has drawn much attention of the control community.

Systems with time delay under deterministic excitation have been studied by many researchers (Agrawal and Yang, 1997; Atay, 1998; Hu and Wang, 2002; Kuo, 1987; Malek-Zavarei and Jamshidi, 1987; Pu, 1998; Stepan, 1989). The study on those systems under stochastic excitation is very limited. A linearly controlled system with deterministic and random time delays excited by Gaussian white noise was treated by Grigoriu (1997) and the stability of such a system was investigated by means of the Lyapunov exponent. The effects of time delay on the controlled linear systems under Gaussian random excitation were studied by Di Paola and Pirrotta (2001) using an approach based on the Taylor expansion of the control force and another approach to find an exact stationary solution. The stochastic averaging method for quasi-integrable Hamiltonian systems with time-delayed feedback control has been proposed by the present authors and the effects of the time delay on the system response and stability were studied in Liu and Zhu (2007, 2008), Zhu and Liu (2007).

Hopf bifurcation often occurs in time-delayed deterministic systems and has been studied by using the linear stability analysis method (Hassard *et al.*, 1981; Kuznetstov, 1998), the invariant manifold reduction and the normal form method (Kalmar-Nagy *et al.*, 2001; Xu and Chung, 2003), the averaging method (Stephen and Richard, 2002), the multiple scales method (Das and Chatterjee, 2002). Much work has been done on the effect of noise on the bifurcation (Arnold *et al.*, 1996; Srinamachchivaya, 1990). A procedure was proposed for analysing the stochastic Hopf bifurcation of quasi-nonintegrable Hamiltonian systems (Zhu and Huang, 1999). The study on stochastic Hopf bifurcation of excitation systems with time delay is very limited. Longtin (1991) studied the influence of coloured noise on the Hopf bifurcation in a first-order delay differential equation of nonlinear delayed feedback systems.

In the present paper, the stochastic Hopf bifurcation of quasi-integrable Hamiltonian systems with time-delayed feedback control is studied. First, the stochastic averaging method for quasi-integrable Hamiltonian systems with time-delayed feedback control is introduced. The time-delayed feedback control forces are expressed in terms of the system states without time delay in the average sense. The equations of the system are reduced to a set of averaged Itô stochastic differential equations by applying the stochastic averaging method for quasi-integrable Hamiltonian systems. Then, the expression for the average bifurcation parameter of the averaged system is obtained and a criterion for determining the stochastic Hopf bifurcation caused by the time-delayed feedback control forces in the original system by using the average bifurcation parameter is proposed. An example is given to illustrate the proposed criterion and the results of numerical simulation are obtained to verify the effectiveness of the proposed criterion. The effect of time delay in the control forces on the stochastic Hopf bifurcation is analysed in detail.

2. Stochastic averaging of quasi-integrable Hamiltonian systems with time-delayed feedback control

Consider an n-DOF quasi-Hamiltonian system with time-delayed feedback control governed by the following Itô stochastic differential equations

$$\begin{aligned} dQ_i &= \frac{\partial H'}{\partial P_i} \\ dP_i &= -\Big[\frac{\partial H'}{\partial Q_i} + \varepsilon c'_{ij}\frac{\partial H'}{\partial P_i} + \varepsilon F_i(\mathbf{Q}_\tau, \mathbf{P}_\tau)\Big]dt + \sqrt{\varepsilon}\sigma_{ik} dB_k(t) \\ i,j &= 1,2,\dots,n \qquad\qquad k = 1,2,\dots,m \end{aligned} \tag{2.1}$$

where Q_i and P_i are generalized displacements and momenta, respectively, $\mathbf{Q} = [Q_1, Q_2, \dots, Q_n]^\top$, $\mathbf{P} = [P_1, P_2, \dots, P_n]^\top$; $H' = H'(\mathbf{Q}, \mathbf{P})$ is twice differentiable Hamiltonian; ε is a small positive parameter; $\varepsilon c'_{ij} = \varepsilon c'_{ij}(\mathbf{Q}, \mathbf{P})$ represent the coefficients of quasi linear damping; $B_k(t)$ are standard Wiener processes and $\sqrt{\varepsilon}\sigma_{ik}$ represent amplitudes of stochastic excitations; $\varepsilon F_i(\mathbf{Q}_\tau, \mathbf{P}_\tau)$ with $\mathbf{Q}_\tau = \mathbf{Q}(t-\tau)$ and $\mathbf{P}_\tau = \mathbf{P}(t-\tau)$ denote the time-delayed feedback control forces, τ is the time delay, and $\varepsilon F_i(\mathbf{Q}_\tau, \mathbf{P}_\tau) = 0$ when $t \in [0, \tau]$.

Assume that the Hamiltonian H' associated with system (2.1) is separable and of the form

$$H' = \sum_{i=1}^{n} H'_i(q_i, p_i) \qquad\qquad H'_i = \frac{1}{2}p_i^2 + G(q_i) \tag{2.2}$$

where $G(q_i) \geqslant 0$ is symmetric with respect to $q_i = 0$, and with minimum at $q_i = 0$, i.e., the Hamiltonian system with Hamiltonian H' is integrable and has a family of periodic solutions around the origin. When ε is small, the solution to Eq. (2.1) is of the form (Huang *et al.*, 2000; Zhu *et al.*, 2003)

$$\begin{aligned} Q_i(t) &= A_i \cos\Phi_i(t) \qquad\qquad P_i(t) = -A_i\frac{d\Theta_i}{dt}\sin\Phi_i(t) \\ \Phi_i(t) &= \Theta_i(t) + \Gamma_i(t) \end{aligned} \tag{2.3}$$

where $\cos\Phi(t)$ and $\sin\Phi(t)$ are called generalized harmonic functions. For quasi-integrable Hamiltonian systems, $A_i(t)$ and $\Gamma_i(t)$ are slowly varying processes and the averaged value of the instantaneous frequency $d\Theta_i/dt$ is equal to $\omega_i(A_i)$. For a small delay time τ, we have the following approximate expressions for time-delayed state variables

$$
\begin{aligned}
Q_i(t-\tau) &= A_i(t-\tau)\cos\Phi_i(t-\tau) \approx A_i(t)\cos[\omega_i(t-\tau)+\Gamma_i(t)] = \\
&= Q_i(t)\cos\omega_i\tau - \frac{P_i}{\omega_i}\sin\omega_i\tau \\
P_i(t-\tau) &= -A_i(t-\tau)\frac{d\Theta_i(t-\tau)}{dt}\sin\Phi_i(t-\tau) \approx \\
&\approx -A_i(t)\omega_i\sin[\omega_i(t-\tau)+\Gamma_i(t)] = P_i\cos\omega_i\tau + Q_i(t)\omega_i\sin\omega_i\tau
\end{aligned}
\tag{2.4}
$$

Thus, the time-delayed feedback control forces $\varepsilon F_i(\mathbf{Q}_\tau, \mathbf{P}_\tau)$ can be approximately expressed in terms of system state variables without the time delay. Note that the numerical results in the present paper and in Liu and Zhu (2007, 2008), Zhu and Liu (2007) show that Eqs. (2.4) holds even for larger τ.

The terms $\varepsilon F(\mathbf{Q}_\tau, \mathbf{P}_\tau)$ in Eqs. (2.1) can be split into two parts: one has the effect of modifying the conservative forces and the other modifying the damping forces. The first part can be combined with $-\partial H'/\partial Q_i$ to form overall effective conservative forces $-\partial H/\partial Q_i$ with a new Hamiltonian $H = H(\mathbf{Q}, \mathbf{P}; \tau)$ and with $\partial H/\partial P_i = \partial H'/\partial P_i$. The second part may be combined with $-\varepsilon c'_{ij}\partial H'/\partial P_j$ to constitute effective damping forces $-\varepsilon m_{ij}\partial H/\partial P_i$ with $m_{ij} = m_{ij}(\mathbf{Q}, \mathbf{P}; \tau)$. With these accomplished, Eqs. (2.1) can be rewritten as

$$
\begin{aligned}
dQ_i &= \frac{\partial H}{\partial P_i}dt \\
dP_i &= -\Big(\frac{\partial H}{\partial Q_i} + \varepsilon m_{ij}\frac{\partial H}{\partial P_j}\Big)dt + \sqrt{\varepsilon}\sigma_{ik}dB_k(t) \\
i,j &= 1,2,\dots,n \qquad\qquad k = 1,2,\dots,m
\end{aligned}
\tag{2.5}
$$

where $H = H(\mathbf{Q}, \mathbf{P}; \tau)$, $m_{ij} = m_{ij}(\mathbf{Q}, \mathbf{P}; \tau)$. Eqs. (2.5) is the Itô equations for quasi-integrable Hamiltonian systems without time delay.

Assume that the Hamiltonian system with Hamiltonian H is still integrable and nonresonant. That is, the Hamiltonian system has n independent first integrals $H_1, H_2, \dots, H_n$, which are in involution. The term "in involution" implies that the Poisson bracket of any two of $H_1, H_2, \dots, H_n$ vanishes. In principle, n pairs of action-angle variables I_i, θ_i can be introduced for an integrable Hamiltonian system of n-DOF. Nonresonance means that the n frequencies $\omega_i = d\theta_i/dt$ do not satisfy the following resonant relation

$$k_i^u \omega_i = 0(\varepsilon) \tag{2.6}$$

where k_i^u are integers.

Introduce transformations

$$H_r^\varepsilon = H_r(\mathbf{Q}, \mathbf{P}, \varepsilon) \qquad r = 1, 2, \ldots, n \tag{2.7}$$

The Itô stochastic differential equations for H_r^ε are obtained from Eqs. (2.5) by using the Itô differential rule as follows

$$\begin{aligned} dH_r^\varepsilon &= \varepsilon\Big(-m_{ij}\frac{\partial H}{\partial P_j}\frac{\partial H_r^\varepsilon}{\partial P_i} + \frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 H_r^\varepsilon}{\partial P_i \partial P_j}\Big)dt + \sqrt{\varepsilon}\sigma_{ik}\frac{\partial H_r^\varepsilon}{\partial P_i}dB_k(t) \\ r, i, j &= 1, 2, \ldots, n \qquad k = 1, 2, \ldots, m \end{aligned} \tag{2.8}$$

where P_i are replaced by H_r^ε in terms of Eq. (2.7). It is seen from Eqs. (2.5) and (2.7) that Q_i are rapidly varying processes while H_r^ε are slowly varying processes. According to the Khasminskii theorem (Khasminskii, 1967), $\mathbf{H}^\varepsilon = [H_1^\varepsilon, H_2^\varepsilon, \ldots, H_n^\varepsilon]^\top$ converges weakly to an n-dimensional vector diffusion process $\mathbf{H} = [H_1, H_2, \ldots, H_n]^\top$ in a time interval $O(\varepsilon^{-1})$ as $\varepsilon \to 0$. For each bounded and continuous real-valued function $f(\mathbf{H})$, the expression "H_r^ε converges weakly to H_r" means $\int f(\mathbf{H})\, dP^\varepsilon(\mathbf{H}) \to \int f(\mathbf{H})\, dP(\mathbf{H})$ as $\varepsilon \to 0$, where $P^\varepsilon(\mathbf{H})$ and $P(\mathbf{H})$ are, respectively, the joint probability distributions of $\mathbf{H}^\varepsilon$ and $\mathbf{H}$. The error between the solutions of the original and averaged systems is of the order ε.

The Itô stochastic differential equations for this n-dimensional vector diffusion process can be obtained by applying the time averaging to Eq. (2.8). The result is

$$dH_r = a_r(\mathbf{H})dt + \overline{\sigma}_{rk}(\mathbf{H})d\overline{B}_k(t) \qquad \begin{aligned} r &= 1, 2, \ldots, n \\ k &= 1, 2, \ldots, m \end{aligned} \tag{2.9}$$

where $\mathbf{H} = [H_1, H_2, \ldots, H_n]^\top$; $\overline{B}_k(t)$ are independent unit Wiener processes

$$\begin{aligned} a_r(\mathbf{H}) &= \varepsilon\Big\langle -m_{ij}\frac{\partial H}{\partial P_j}\frac{\partial H_r}{\partial P_i} + \frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 H_r}{\partial P_i \partial P_j}\Big\rangle_t \\ b_{rs}(\mathbf{H}) &= \overline{\sigma}_{rk}(\mathbf{H})\overline{\sigma}_{sk}(\mathbf{H}) = \varepsilon\Big\langle \sigma_{ik}\sigma_{jk}\frac{\partial H_r}{\partial P_i}\frac{\partial H_s}{\partial P_j}\Big\rangle_t \end{aligned} \tag{2.10}$$

$$\langle[\cdot]\rangle_t = \lim_{T\to\infty}\frac{1}{T}\int\limits_T^{t_0+T}[\cdot]\,dt$$

Note that H_r are kept constant in performing the time averaging.

The time averaging in Eqs. (2.10) may be replaced by space averaging. For example, suppose that the Hamiltonian is separable and equal to the sum of n independent first integers, i.e.

$$H(\boldsymbol{q},\boldsymbol{p}) = \sum_{r=1}^{n} H_r(q_r,p_r) \tag{2.11}$$

and for each H_r there is a periodic orbit with the period T_r. Then, the averaged drift and diffusion coefficients in Eqs. (2.10) become

$$\begin{aligned} a_r(H) &= \frac{\varepsilon}{T}\oint\Big(-m_{ij}\frac{\partial H}{\partial P_j}\frac{\partial H_r}{\partial P_i} + \frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 H_r}{\partial P_i \partial P_j}\Big)\prod_{u=1}^{n}\Big(\frac{\partial H_u}{\partial P_u}\Big)^{-1}\,dq_u \\ b_{rs}(H) &= \frac{\varepsilon}{T}\oint\Big(\sigma_{ik}\sigma_{jk}\frac{\partial H_r}{\partial P_i}\frac{\partial H_r}{\partial P_j}\Big)\prod_{u=1}^{n}\Big(\frac{\partial H_u}{\partial P_u}\Big)^{-1}\,dq_u \end{aligned} \tag{2.12}$$

where $\oint[\cdot]\prod_{u=1}^{n}(\cdots)\,dq_u$ represents an n-fold loop integral and

$$T = T(\mathbf{H}) = \prod_{u=1}^{n} T_u = \oint \prod_{u=1}^{n}\Big(\frac{\partial H_u}{\partial P_u}\Big)^{-1}\,dq_u \tag{2.13}$$

Note that averaged Eq. (2.9) is much simpler than original Eqs. (2.5). The dimension of the former equation is only a half of that of the later equation. Averaged equation (2.9) contains only slowly varying process, while Eqs. (2.5) contains both rapidly and slowly varying processes. Furthermore, the averaged equation can be used to study the long-term behaviour of the system, such as stability, stationary response and first-passage failure, since the convergence of H_r^{ε} to the diffusion process holds even for $t \to \infty$ (Blankenship and Papanicolaou, 1978; Kushner, 1984).

3. Determination of Hopf bifurcation by the average bifurcation parameter

It is well known that for a Duffing-van der Pol oscillator under parametric excitation of Gaussian white noise, stochastic Hopf bifurcation consists of a dynamical bifurcation (D-bifurcation) and a phenomenological bifurcation (P-bifurcation) (Arnold *et al.*, 1996). Before the D-bifurcation, the trivial solution is asymptotically stable with probability one and the stationary joint probability density of the displacement and velocity is the Dirac delta function. The D-bifurcation occurs when the largest Lyapunov exponent vanishes. After the D-bifurcation and before P-bifurcation, the trivial solution is unstable and the stationary joint probability density of the displacement and velocity is unimodel with the peak at the origin. After the P-bifurcation, the trivial solution is still unstable, while the stationary joint probability density of the displacement and velocity becomes crater-like. The interval between the D-bifurcation and P-bifurcation is called the bifurcation interval.

The analysis of stochastic Hopf bifurcation can be greatly simplified by using the stochastic averaging method for quasi Hamiltonian systems. By using the relationship between the one-demensional diffusion process and its boundaries, Zhu and Huang (1999) proposed a criterion for determining stochastic Hopf bifurcation (both D-bifurcation and P-bifurcation) in quasi non-integrable Hamiltonian systems using the diffusion exponent, draft exponent and a character value. In the following, we will generalize this criterion to quasi-integrable Hamiltonian systems with time-delayed feedback control.

Based on Eq. (2.11), we introduce the following new variable

$$\alpha_r = \frac{H_r}{H} \qquad r = 1, 2, \ldots, n \tag{3.1}$$

Note that $\sum_{r=1}^{n} \alpha_r = 1$, so only $n-1$ variables for α_r in Eq. (3.1) are independent. In the following, we take the first $n-1$ variables for $\boldsymbol{a}' = [\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$ as independent variables with α_n replaced by $\alpha_n = 1 - \sum_{r=1}^{n-1} \alpha_r$. The Itô equations for H and α_r can be obtained from Eq. (2.9) by using the Itô differential rule as follows

$$\begin{aligned} dH &= Q(\boldsymbol{a}', H; \tau)dt + \varSigma_k(\boldsymbol{a}', H; \tau)d\overline{B}_k(t) \\ d\alpha_r &= m_r(\boldsymbol{a}', H; \tau)dt + \widetilde{\sigma}_{rk}(\boldsymbol{a}', H; \tau)d\overline{B}_k(t) \\ r &= 1, 2, \ldots, n-1 \qquad k = 1, 2, \ldots, m \end{aligned} \tag{3.2}$$

where

$$Q(\boldsymbol{a}', H; \tau) = \sum_{r=1}^{n} a_r(\boldsymbol{a}', H; \tau)$$

$$\varSigma_k(\boldsymbol{a}', H; \tau) = \sum_{r=1}^{n} \overline{\sigma}_{rk}(\boldsymbol{a}', H; \tau)$$

$$
\begin{aligned}
m_r(\boldsymbol{a}', H; \tau) &= -\alpha_r \sum_{s=1}^{n} \frac{1}{H} a_s(\boldsymbol{a}', H; \tau) + \\
&\quad -\frac{1}{2} \sum_{s=1}^{n} \sum_{k=1}^{m} \frac{1}{H^2} \overline{\sigma}_{rk}(\boldsymbol{a}', H; \tau) \overline{\sigma}_{sk}(\boldsymbol{a}', H; \tau) + \\
&\quad +\frac{1}{2} \alpha_r \sum_{s,s'=1}^{n} \sum_{k=1}^{m} \frac{1}{H^2} \overline{\sigma}_{sk}(\boldsymbol{a}', H; \tau) \overline{\sigma}_{s'k}(\boldsymbol{a}', H; \tau) + \frac{1}{H} a_r(\boldsymbol{a}', H; \tau) \\
\widetilde{\sigma}_{rk}(\boldsymbol{a}', H; \tau) &= \frac{1}{H} \overline{\sigma}_{rk}(\boldsymbol{a}', H; \tau) - \alpha_r \sum_{s=1}^{n} \frac{1}{H} \overline{\sigma}_{sk}(\boldsymbol{a}', H; \tau)
\end{aligned}
\tag{3.3}
$$

For the one-dimensional diffusion process $H(t)$ governed by Eq. $(3.2)_1$, the boundary $H \to \infty$ must be either an entrance or a repulsively natural in order that the trivial solution $H = 0$ is stable in probability or $H(t)$ has a stationary probability density, i.e., the boundary $H \to \infty$ must be either an entrance or repulsively natural during the first and second bifurcation. In the following, we will focus our attention on the qualitative change in sample behaviour of $H(t)$ near the boundary $H = 0$ during the first and second bifurcation.

For the one-dimensional diffusion process reduced from higher-dimensional systems undergoing parametric excitations by using the stochastic averaging, the boundaries $H = 0$ and $H \to \infty$ are often singular and the sample behaviour of the process near the boundaries are characterised by the diffusion exponent, the drift exponent and the character value (Lin and Cai, 1995). For a singular left boundary of the first kind, i.e., $\varSigma_k(\boldsymbol{a}, 0; \tau) = 0$, the diffusion exponent α_l, the drift exponent β_l and the character value c_l are defined as follows

$$
\begin{aligned}
& b'(\boldsymbol{a}', H; \tau) = (\varSigma_k(\boldsymbol{a}', H; \tau))^2 = O(H^{\alpha_l}) && \alpha_l > 0 \quad \text{as} \quad H \to 0 \\
& Q(\boldsymbol{a}', H; \tau) = O(H^{\beta_l}) && \beta_l > 0 \quad \text{as} \quad H \to 0 \\
& c_l(\boldsymbol{a}'; \tau) = \lim_{H \to 0^+} \frac{2Q(\boldsymbol{a}', H; \tau) H^{\alpha_l - \beta_l}}{b'(\boldsymbol{a}', H; \tau)}
\end{aligned}
\tag{3.4}
$$

where $O(\cdot)$ denotes the order of magnitude of $(\cdot)$. For a singular right boundary of the second kind, i.e., $m(\infty) \to \infty$, the diffusion exponent α_r, the drift exponent β_r and the character value c_r are defined as follows

$$
\begin{aligned}
& b'(\boldsymbol{a}', H; \tau) = (\varSigma_k(\boldsymbol{a}', H; \tau))^2 = O(H^{\alpha_r}) && \alpha_r > 0 \quad \text{as} \quad H \to \infty \\
& Q(\boldsymbol{a}', H; \tau) = O(H^{\beta_r}) && \beta_r > 0 \quad \text{as} \quad H \to \infty \\
& c_r(\boldsymbol{a}'; \tau) = \lim_{H \to +\infty} \frac{2Q(\boldsymbol{a}', H; \tau) H^{\alpha_r - \beta_r}}{b'(\boldsymbol{a}', H; \tau)}
\end{aligned}
\tag{3.5}
$$

The criteria for classification of the singular boundary based on values of the diffusion exponent, the drift exponent and the character value can be found in Tables in Lin and Cai (1995).

Considering Eqs. (3.4), one can obtain the following asymptotic expression for the stationary probability density of $H(t)$

$$p(H;\tau) = O\Big\{H^{-\alpha_l} \exp\Big[c_l \int\limits_0^H x^{(\beta_l-\alpha_l)}\, dx\Big]\Big\} \qquad \text{as} \quad H \to 0 \tag{3.6}$$

Two cases can be identified.

Case 1. $\beta_l - \alpha_l = -1$. In this case

$$p(H;\tau) = O(H^v) \qquad \text{as} \quad H \to 0 \tag{3.7}$$

with

$$v(\boldsymbol{a}';\tau) = c_l(\boldsymbol{a}';\tau) - \alpha_l \tag{3.8}$$

Particularly, when $\beta_l = 1$ and $\alpha_l = 2$, the diffusion and drift coefficients in Eq. $(3.2)_2$ are linear. Introduce an average bifurcation parameter $\overline{v}(\tau)$ defined by

$$\overline{v}(\tau) = \int\limits_{\Omega} v(\boldsymbol{a}';\tau)p(\boldsymbol{a}';\tau)\, d\boldsymbol{a}' \qquad\qquad \Omega = \Big\{\boldsymbol{a}'\Big|\sum_{r=1}^{n}\alpha_i = 1,\ 0 \leqslant \alpha_i \leqslant 1\Big\} \tag{3.9}$$

where $p(\boldsymbol{a}';\tau)$ is the stationary solution to the Fokker-Plank-Kolmogorov (FPK) equation associated with the Itô differential equations in Eq. $(3.2)_2$.

Equation (3.7) is non-integrable and the probability density $p(H;\tau)$ is the delta function if $\overline{v}(\tau) < -1$. When $-1 < \overline{v}(\tau) < 0$, Eq. (3.7) is integrable and a stationary probability density $p(H;\tau)$ exists with a peak at $H = 0$. If $\overline{v}(\tau) > 0$, then Eq. (3.7) is integrable and $p(H;\tau)$ exists with a peak away from $H = 0$. Thus, the first bifurcation (D-bifurcation) occurs at $\overline{v}(\tau) = -1$ and the second bifurcation (P-bifurcation) at $\overline{v}(\tau) = 0$ provided that the right boundary $H \to \infty$ is an entrance or repulsively natural. It is interesting to note that the condition for the first bifurcation here is consistent with that obtained from the necessary and sufficient condition for the asymptotic stability with probability one of the trivial solution.

Case 2. $\beta_l - \alpha_l \neq -1$. In this case

$$p(H) = O\Big\{H^{-\alpha_l} \exp\Big[\frac{c_l}{1+\beta_l-\alpha_l} H^{(\beta_l-\alpha_l+1)}\Big]\Big\} \qquad \text{as} \quad H \to 0 \tag{3.10}$$

which cannot be expressed in the form of Eq. (3.7). It can be shown that it is impossible for $p(H;\tau)$ to have a peak at or near $H=0$ when it exists. In other words, in this case, although the first bifurcation may occur, it is impossible for the second bifurcation to occur.

The D-bifurcation and P-bifurcation of $H(t)$ implies a stochastic Hopf bifurcation of original system (2.1). In other words, the stochastic Hopf bifurcation of a quasi-integrable Hamiltonian system with time-delayed feedback control can be determined by examining the sample behaviour of the one-dimensional averaged diffusion process at its boundaries $H=0$ and $H\to\infty$. The first bifurcation occurs when $\overline{v}(\tau)=-1$, and the second bifurcation occurs when $\overline{v}(\tau)=0$ while the right boundary is either an entrance or repulsively natural.

4. Example

To illustrate the above criterion for stochastic Hopf bifurcation, consider two coupled Rayleigh oscillators with time-delayed feedback control subject to parametric excitations of Gaussian white noise. The equations of motion of the system are of the form

$$\begin{aligned}
&\ddot{X}_1+\Big(-\beta'_{10}+\beta_{11}\dot{X}_1^2+\beta_{12}\dot{X}_2^2\Big)\dot{X}_1+{\omega'_1}^2X_1=\\
&\qquad=-\eta_1\dot{X}_{1\tau}+f_{11}\dot{X}_1W_1(t)+f_{12}\dot{X}_2W_2(t)\\
&\ddot{X}_2+\Big(-\beta'_{20}+\beta_{21}\dot{X}_1^2+\beta_{22}\dot{X}_2^2\Big)\dot{X}_2+{\omega'_2}^2X_2=\\
&\qquad=-\eta_2\dot{X}_{2\tau}+f_{21}\dot{X}_1W_1(t)+f_{22}\dot{X}_2W_2(t)
\end{aligned} \tag{4.1}$$

where X_i are generalized coordinates; β'_{i0} and β_{ij} $(i,j=1,2)$ are damping coefficients; ω'_i are natural frequencies of the two linear oscillators; $W_k(t)$ $(k=1,2)$ are independent Gaussian white noises with intensities $2D_{kk}$; $-\eta_i\dot{X}_{i\tau}$ represent the time-delayed feedback control forces. Here we study the effects of time delay in the feedback control forces on the stochastic Hopf bifurcation of system (4.1).

Following Eqs. (2.4), the time-delayed feedback control forces can be expressed in terms of system state variables without time delay as follows

$$-\eta_i\dot{X}_{i\tau}=-\eta_i\dot{X}_i\cos\omega'_i\tau-\eta_i\omega'_iX_i\sin\omega'_i\tau \qquad i=1,2 \tag{4.2}$$

On the right hand side of Eq. (4.2), the first terms are dissipative, while the second terms are conservative. They can be combined, respectively, with the damping terms and conservative terms of Eq. (4.1) to constitute effective damping terms and effective conservative terms.

Let $X_1 = Q_1$, $X_2 = Q_2$, $\dot{X}_1 = P_1$, $\dot{X}_2 = P_2$. By applying the stochastic averaging method for quasi-integrable Hamiltonian systems to modified Eq. (4.1), the following averaged Itô equations can be obtained in the nonresonant case

$$\begin{aligned} dH_1 &= \Big[(\beta_{10} + 2f_{11}^2 D_{11})H_1 - \frac{3}{2}\beta_{11}H_1^2 - \beta_{12}H_1H_2 + f_{12}^2 D_{22}H_2\Big]dt + \\ &\quad +\sigma_{11}dB_1(t) + \sigma_{12}dB_2(t) \\ dH_2 &= \Big[(\beta_{20} + 2f_{22}^2 D_{22})H_2 - \frac{3}{2}\beta_{22}H_2^2 - \beta_{21}H_1H_2 + f_{21}^2 D_{11}H_1\Big]dt + \\ &\quad +\sigma_{21}dB_1(t) + \sigma_{22}dB_2(t) \end{aligned} \tag{4.3}$$

with

$$\begin{aligned} H_i &= \frac{1}{2}(P_i^2 + \omega_i^2 Q_i^2) \qquad\qquad \omega_i^2 = \omega_i'^2 + \eta_i\omega_i' \sin\omega_i'\tau \\ \beta_{i0} &= \beta_{i0}' + \eta_i \cos omega_i'\tau \\ b_{11} &= \sigma_{1j}\sigma_{1j} = 3D_{11}f_{11}^2H_1^2 + 2f_{12}^2D_{22}H_1H_2 \\ b_{22} &= \sigma_{2j}\sigma_{2j} = 3D_{22}f_{22}^2H_2^2 + 2f_{21}^2D_{11}H_1H_2 \\ b_{12} &= b_{21} = \sigma_{1j}\sigma_{2j} = 0 \end{aligned} \tag{4.4}$$

The Itô differential equations associated with $H = H_1 + H_2$ and $\alpha_1 = H_1/H$ can be obtained by using the Itô differential rule as follows

$$\begin{aligned} dH &= (Q_1H + Q_2H^2)dt + \Sigma_1 H dB_1(t) \\ d\alpha_1 &= m_1dt + \widetilde{\sigma}_1 dB_1(t) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} Q_1 &= (\beta_{10} + 2f_{11}^2D_{11} + f_{21}^2D_{11})\alpha_1 + (\beta_{20} + 2f_{22}^2D_{22} + f_{12}^2D_{22})(1-\alpha_1) \\ Q_2 &= -\frac{3}{2}\beta_{11}\alpha_1^2 - \frac{3}{2}\beta_{22}(1-\alpha_1)^2 - (\beta_{12}+\beta_{21})\alpha_1(1-\alpha_1) \\ \Sigma_1^2 &= 3f_{11}^2D_{11}\alpha_1^2 + 3f_{22}^2D_{22}(1-\alpha_1)^2 + 2(f_{21}^2D_{11} + f_{12}^2D_{22})\alpha_1(1-\alpha_1) \\ m_1 &= \Big(\frac{1}{2} - \alpha_1\Big)\varphi(\alpha_1) + 2\alpha_1(1-\alpha_1)(\lambda_1 - \lambda_2) \end{aligned} \tag{4.6}$$

$$\tilde{\sigma}_1^2 = \alpha_1(1-\alpha_1)\varphi(\alpha_1) \qquad \varphi(\alpha_1) = a\alpha_1^2 + b\alpha_1 + c \qquad c = G_{12}$$

$$a = G_{12} + G_{21} - G_{11} - G_{22} \qquad b = G_{11} + G_{22} - 2G_{12} \qquad G_{11} = 3f_{11}^2 D_{11}$$

$$\lambda_1 = \frac{1}{2}\beta_{10} + \frac{1}{4}f_{11}^2 D_{11} \qquad \lambda_2 = \tfrac{1}{2}\beta_{20} + \tfrac{1}{4}f_{22}^2 D_{22} \qquad G_{22} = 3f_{22}^2 D_{22}$$

$$G_{12} = 2f_{12}^2 D_{22} \qquad G_{21} = 2f_{21}^2 D_{11}$$

For $H(t)$ governed by Eqs. (4.5) at $H \to \infty$, the diffusion exponent $\alpha_r = 2$, the drift exponent $\beta_r = 2$. If $\beta_{ij} > 0$ $(i,j = 1,2)$, then $Q_2 < 0$, the boundary $H \to \infty$ is an entrance. At the boundary $H = 0$, the diffusion exponent, the draft exponent and the character value are

$$\begin{aligned} &\alpha_l = 2 \qquad \beta_l = 1 \qquad c_l = \frac{2Q_1}{\Sigma_1^2} = c_l(\alpha_1;\tau) \\ &v(\alpha_1;\tau) = c_l(\alpha_1;\tau) - 2 \end{aligned} \tag{4.7}$$

The stationary solution $p(\alpha_1;\tau)$ to the FPK equation associated with Itô Eqs. (4.6) is

$$p(\alpha_1;\tau) = \frac{C}{\varphi(\alpha_1)} F(\alpha_1) \tag{4.8}$$

where

$$\begin{aligned} F(\alpha_1) &= \begin{cases} \exp\Big(\dfrac{4(\lambda_1-\lambda_2)}{\sqrt{\Delta}} \ln\Big|\dfrac{2a\alpha_1 + b - \sqrt{\Delta}}{2a\alpha_1 + b + \sqrt{\Delta}}\Big|\Big) & \text{for} \quad \Delta > 0 \\ \exp\Big(\dfrac{8(\lambda_1-\lambda_2)}{\sqrt{-\Delta}} \arctan\Big|\dfrac{2a\alpha_1 + b}{\sqrt{-\Delta}}\Big|\Big) & \text{for} \quad \Delta < 0 \\ \exp\Big(\dfrac{8(\lambda_1-\lambda_2)}{2a\alpha_1 + b}\Big) & \text{for} \quad \Delta = 0 \end{cases} \\ C &= \frac{4(\lambda_1-\lambda_2)}{F(1) - F(0)} \qquad \Delta = b^2 - 4ac \end{aligned} \tag{4.9}$$

The average bifurcation parameter $\overline{v}(\tau)$ can be obtained as follows

$$\overline{v}(\tau) = \int_0^1 v(\alpha_1;\tau) p(\alpha_1;\tau)\, d\alpha_1 \tag{4.10}$$

The stochastic Hopf bifurcation of system (4.1) can be determined by using the average bifurcation parameter $\overline{v}(\tau)$. If $\overline{v}(\tau) < -1$, the probability density

$p(H;\tau)$ is the delta function and the system is stable; if $-1 < \overline{v}(\tau) < 0$, the probability density $p(H;\tau)$ exists with a peak at $H = 0$. If $\overline{v}(\tau) > 0$, the probability density $p(H;\tau)$ exists with a peak away from $H = 0$. Thus, the first bifurcation (D-bifurcation) occurs at $\overline{v}(\tau) = -1$ and the second bifurcation (P-bifurcation) occurs at $\overline{v}(\tau) = 0$.

Some numerical results for the stochastic Hopf bifurcation of system (4.1) caused by the time-delayed feedback control are shown in Figs. 1-5. The stability of system (4.1) is shown in parameter plane $(\beta'_{10}, \beta'_{20})$ in terms of the location of the origin $O(0,0)$ in the plane relative to D-bifurcation and P-bifurcation curves.

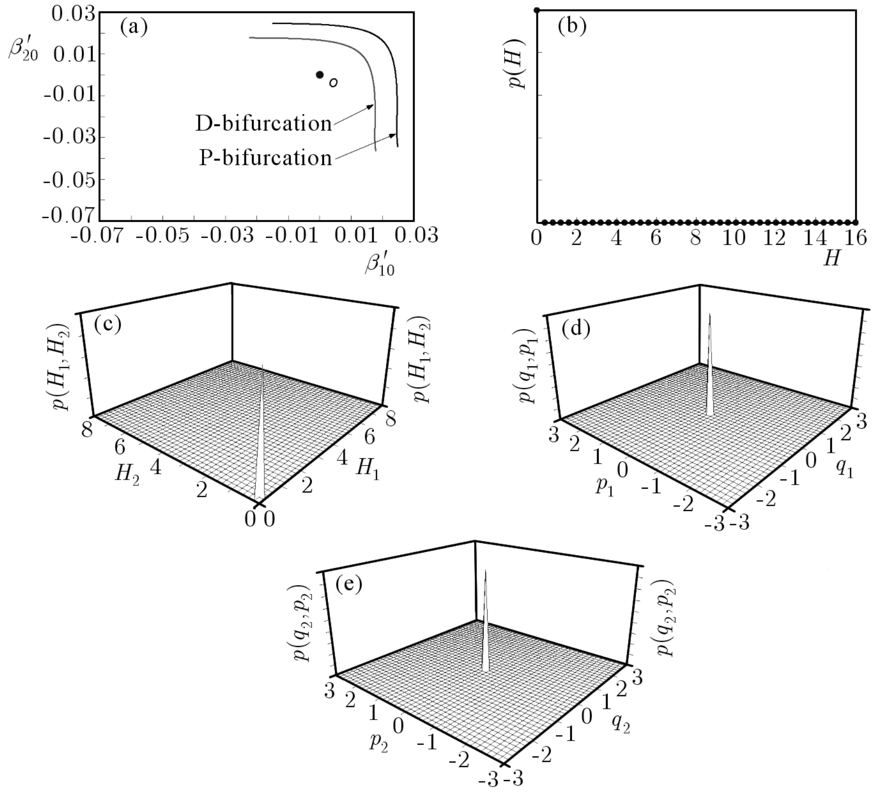

Fig. 1. Results for $\tau = 0$. (a) D-bifurcation and P-bifurcation curves and point $O(0,0)$ in plane $(\beta'_{10}, \beta'_{20})$. (b) Stationary probability density $p(H)$ at point $O(0,0)$. (c) Stationary probability density $p(H_1, H_2)$ at point $O(0,0)$. (d) Stationary probability density $p(q_1, p_1)$ of the first oscillator at point $O(0,0)$. (e) Stationary probability density $p(q_2, p_2)$ of the second oscillator at point $O(0,0)$. The parameters are: $\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0.005$, $\omega'_1 = 1.0$, $\omega'_2 = 1.414$, $2D_1 = 0.01$, $2D_2 = 0.01$, $\eta_1 = \eta_2 = 0.02$, $f_{11} = f_{12} = f_{21} = f_{22} = 1$

The result for $\tau = 0$ is shown in Fig. 1. It is seen that without the time delay, system (4.1) is stable and the stationary probability densities $p(H)$, $p(H_1, H_2)$, $p(q_1, p_1)$ and $p(q_2, p_2)$ are all Dirac delta functions.

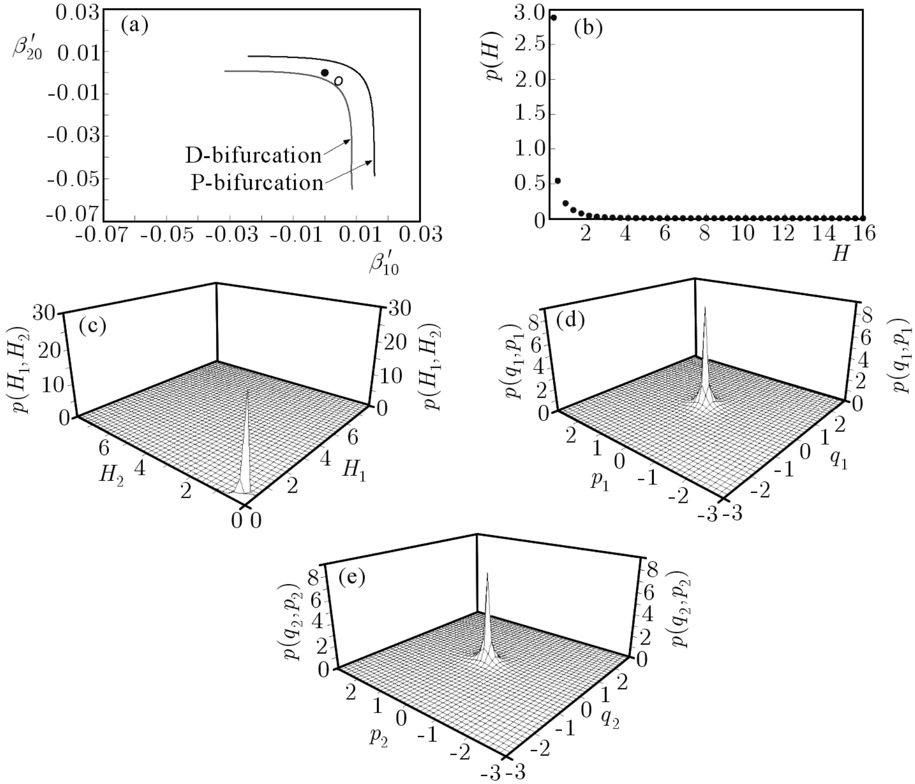

Fig. 2. Results for $\tau = 1.0$. (a) D-bifurcation and P-bifurcation curves and point $O(0,0)$ in plane $(\beta'_{10}, \beta'_{20})$. (b) Stationary probability density $p(H)$ at point $O(0,0)$. (c) Stationary probability density $p(H_1, H_2)$ at point $O(0,0)$. (d) Stationary probability density $p(q_1, p_1)$ of the first oscillator at point $O(0,0)$. (e) Stationary probability density $p(q_2, p_2)$ of the second oscillator at point $O(0,0)$. The parameters are the same as those in Fig. 1

The result for $\tau = 1.0$ is shown in Fig. 2. It is seen that in this case system (4.1) is unstable and the time delay τ is in the bifurcation interval. All the stationary probability densities are normalizable functions with a peak at the origin. It implies that the D-bifurcation occurs in system (4.1) with τ_D value between 0 and 1. This inference is verified by the value $\tau_D = 0.9107$ determined by $\overline{v}(\tau_D) = -1$.

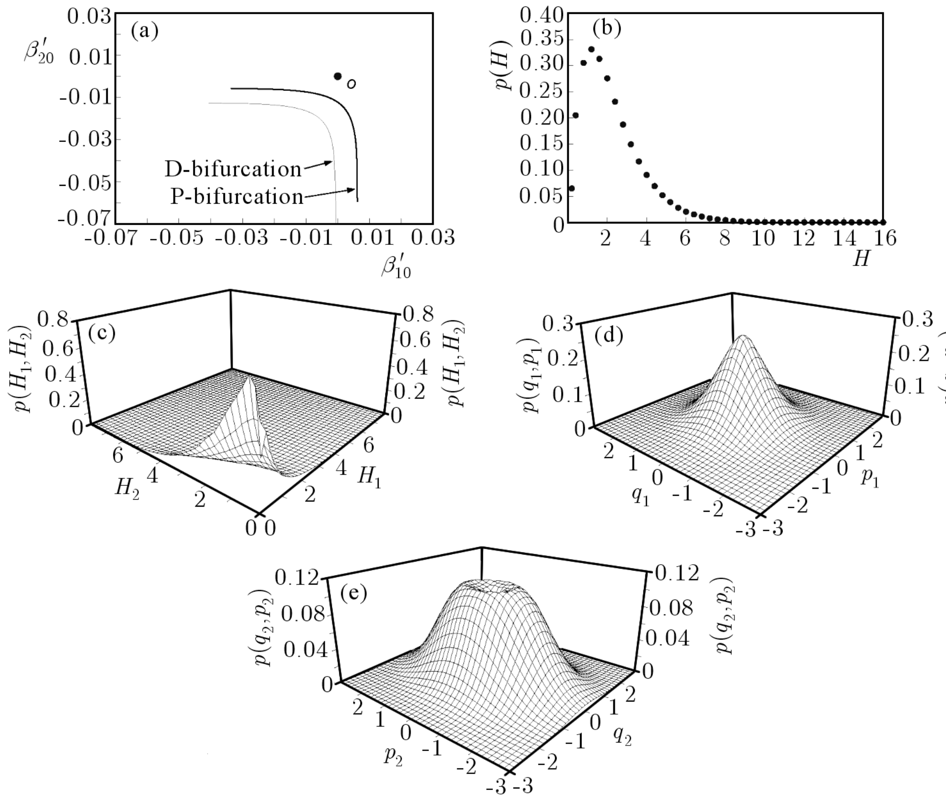

Fig. 3. Results for $\tau = 1.5$. (a) D-bifurcation and P-bifurcation curves and point $O(0,0)$ in plane $(\beta'_{10}, \beta'_{20})$. (b) Stationary probability density $p(H)$ at point $O(0,0)$. (c) Stationary probability density $p(H_1, H_2)$ at point $O(0,0)$. (d) Stationary probability density $p(q_1, p_1)$ of the first oscillator at point $O(0,0)$. (e) Stationary probability density $p(q_2, p_2)$ of the second oscillator at point $O(0,0)$. The parameters are the same as those in Fig. 1

The result for $\tau = 1.5$ is shown in Fig.3. It is seen that system (4.1) is unstable and post D-bifurcation and P-bifurcation. The stationary probability densities $p(H)$ and $p(H_1, H_2)$ are normalizable with their peaks away from the origin and stationary probability density $p(q_2, p_2)$ is crater-like. It implies that the P-bifurcation occurs in the second oscillator of system (4.1) of τ_P value between 1.0 and 1.5. This inference is verified by $\tau_P = 1.1803$ determined by $\overline{v}(\tau_P) = 0$.

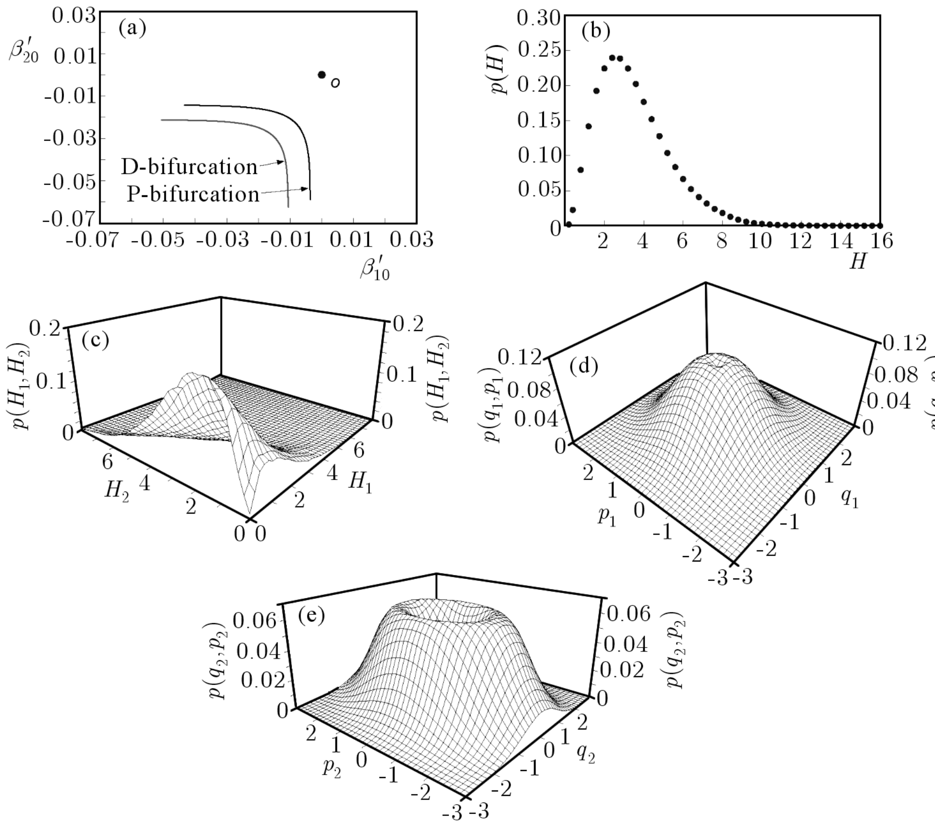

Fig. 4. Results for $\tau = 2.0$. (a) D-bifurcation and P-bifurcation curves and point $O(0,0)$ in plane $(\beta'_{10}, \beta'_{20})$. (b) Stationary probability density $p(H)$ at point $O(0,0)$. (c) Stationary probability density $p(H_1, H_2)$ at point $O(0,0)$. (d) Stationary probability density $p(q_1, p_1)$ of the first oscillator at point $O(0,0)$. (e) Stationary probability density $p(q_2, p_2)$ of the second oscillator at point $O(0,0)$. The parameters are the same as those in Fig. 1

The result for $\tau = 2.0$ is shown in Fig. 4. The difference between Fig. 4 and Fig. 3 is that in this case both stationary probability densities $p(q_1, p_1)$ and $p(q_2, p_2)$ are crater-like, which means both oscillators of system (4.1) are post P-bifurcation. Unfortunately, this second P-bifurcation of system (4.1) can not be predicted by using the criterion proposed in the present method. The result for $\tau = 3.0$ is shown in Fig. 5. System (4.1) is also unstable and post D-bifurcation and P-bifurcation. The stationary probability densities $p(q_1, p_1)$ and $p(q_2, p_2)$ are typical crater-like herein.

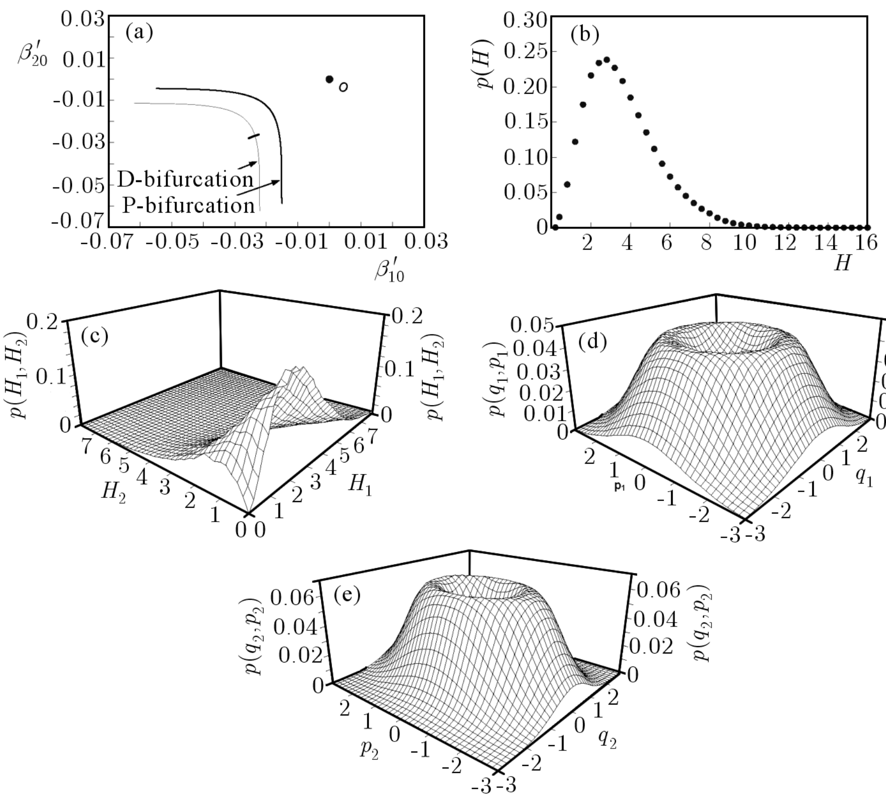

Fig. 5. Results for $\tau = 3.0$. (a) D-bifurcation and P-bifurcation curves and point $O(0,0)$ in plane $(\beta'_{10}, \beta'_{20})$. (b) Stationary probability density $p(H)$ at point $O(0,0)$. (c) Stationary probability density $p(H_1, H_2)$ at point $O(0,0)$. (d) Stationary probability density $p(q_1, p_1)$ of the first oscillator at point $O(0,0)$. (e) Stationary probability density $p(q_2, p_2)$ of the second oscillator at point $O(0,0)$. The parameters are the same as those in Fig. 1

5. Conclusions

In the present paper, a criterion for determining of the stochastic Hopf bifurcation of quasi-integrable Hamiltonian systems with time-delayed feedback control has been proposed based on the stochastic averaging method for quasi-integrable Hamiltonian systems. The time-delayed feedback control forces have been approximately expressed in terms of the system state variables without time delay. The expression for the average bifurcation parameter of the averaged system has been derived. The stochastic Hopf bifurcation caused by the time-delayed feedback control forces in the original system has been examined by using the average bifurcation parameter. The effect of time delay in feedback control on the stochastic Hopf bifurcation has been analysed in detail. The results show that the time delay in the feedback control forces may result in a stochastic Hopf bifurcation in coupled Rayleigh oscillators.

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Stochastyczna bifurkacja Hopfa w quasi-całkowalnych układach Hamiltonowskich sterowanych w pętli sprzężenia zwrotnego z opóźnieniem

Streszczenie

W pracy zajęto się problemem stochastycznej bifurkacji Hopfa quasi-całkowalnych układów Hamiltonowskich o wielu stopniach swobody poddanych wymuszeniu białym szumem z układem sterowania opartym na pętli sprzężenia zwrotnego z opóźnieniem. Najpierw znaleziono przybliżone wyrażenia na siły sterujące w funkcji zmiennych stanu układu bez opóźnienia, a następnie przetransformowano go postaci quasi-całkowalnej, Hamiltonowskiej. Wyprowadzono stochastyczne równania różniczkowe Itô za pomocą metody uśredniania układów quasi-całkowalnych. Znaleziono przybliżoną postać wyrażenia na parametr bifurkacyjny uśrednionego układu i zaproponowano kryterium stwierdzające obecność stochastycznej bifurkacji Hopfa wywołanej siłami sterującymi z opóźnieniem na podstawie wartości zmiany tego parametru. Opracowano szczegółowo przykład do ilustracji działania tego kryterium i zakresu jego stosowalności oraz do prezentacji wpływu opóźnienia w pętli sterownia na stochastyczną bifurkację Hopfa badanego układu.

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