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THE METHOD OF SOLVING POLYNOMIALS IN THE BEAM VIBRATION PROBLEM

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The paper presents a new method of finding an approximate solution to the beam vibration problem. The problem is described with a partial differential equation of the fourth order. Basically, linear differential equations can be solved by means of various methods. The key idea of the presented approach is to find polynomials (solving functions) that satisfy the considered differential equation identically. In this sense, it is a variant of the Trefftz method. The advantage of the method is that the approximate solution (a linear combination of the solving functions) satisfies the equation identically. The initial and boundary conditions are then satisfied approximately. The formulas for solving functions and their derivatives are obtained. The solving Trefftz functions can be used in the whole domain or can be used as base functions in nodeless FEM. Both cases are considered. A numerical example is included.

Key words: beam vibration, Trefftz method, solving functions

1. Introduction

Although the method of solving functions for linear partial differential equations was been developed mainly during the last 10 years, it was thoroughly addressed in the literature. The method was first described by Rosenbloom and Widder (1956) where it was applied to one-dimensional heat conduction problems. In the papers by Ciałkowski *et al.* (1999a,b), Futakiewicz (1999), Futakiewicz and Hożejowski (1998a,b), Futakiewicz *et al.* (1999), Hożewski (1999), Yano *et al.* (1983), the authors described heat functions in different coordinate systems for direct and inverse heat conduction problems. Ciałkowski (1999) considered a highly interesting idea of using heat polynomials as a new type of finite-element base functions.

In another paper Ciałkowski and Frąckowiak (2000) dealt with numerous cases, involving other differential equations such as the Laplace, Poisson, Helmholtz ones. Solving functions for the wave equation are presented in Ciałkowski (2003), Ciałkowski and Frąckowiak (2000, 2003, 2004), Ciałkowski and Jarosławski ((2003), Maciąg (2004, 2005), Maciąg and Wauer (2005a,b). Applications of wave polynomials for elasticity problems are shown in Maciąg (2007b).

In Section 2, solving polynomials and their partial derivatives are considered. Section 3 describes the method of solving functions. An example is discussed in Section 4. Concluding remarks are presented in Section 5.

2. Solving polynomials

Let us consider the equation that describes beam vibration

$$\frac{\partial^4 u}{\partial x^4} + \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{2.1}$$

where $a^2 = EJ/(\rho S)$; E stands for the coefficient of elasticity, J describes the inertia moment of the beam cross section with S being the surface of the beam cross-section. Equation (2.1) describes also vibration of the tuning fork and some problems concerning rotating shafts. In dimensionless coordinates, equation (2.1) has a form

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0 \qquad\qquad x \in (0,1), \quad t > 0 \tag{2.2}$$

Equation (2.1) or (2.2) has to be completed with initial and boundary conditions depending on the type of fixation of the beam.

There are two ways to obtain solving polynomials for equation (2.2). The first one is to expand the function satisfying the equation in power series and reducing some components of the expansion with the use of this equation. The second method consists in using a "generating function". Both ways are equivalent and lead to the same polynomials.

2.1. Expansion of the function satisfying the governing equation into Taylor series

The first way to get solving polynomials for (2.2) is to expand the function that satisfies this equation into Taylor series (comp. Ciałkowski and Frąckowiak, 2000). Let the function $u(x,t)$ satisfy equation (2.2) with given initial and boundary conditions. We assume $u \in C^{N+1}$ in the neighborhood of $(0,0)$. Then, the Taylor series for the function u is

$$u(x,t) = u(0,0) + \frac{du}{1!} + \frac{d^2u}{2!} + \ldots + \frac{d^{n-1}u}{(n-1)!} + R_n \tag{2.3}$$

where

$$d^\nu u = \Big(\frac{\partial u}{\partial x}x + \frac{\partial u}{\partial t}t\Big)^{(\nu)}$$

We denote the k-th derivative of the function $u(x,t)$ in the point $(0,0)$ (i.e. m-th with respect to t and $(k-m)$-th with respect to x, $m = 0,1,\ldots,k$) as $u_{x^{k-m}t^m}$. Hence

$$\frac{d^k u}{k!} = \frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}u_{x^{k-m}t^m}x^{k-m}t^m$$

Using equation (2.2) we can substitute the partial derivative of the order 4ν, $\nu = 1,2,\ldots$ as follows

$$\frac{\partial^{4\nu}u}{\partial x^{4\nu}} = (-1)^\nu\frac{\partial^{2\nu}u}{\partial t^{2\nu}} \tag{2.4}$$

Formula (2.4) holds for differentials of the order greater than 3. For example

$$\frac{du}{1!} = \binom{1}{0}u_x x + \binom{1}{1}u_t t$$

$$\frac{d^2u}{2!} = \binom{2}{0}u_{x^2}\frac{x^2}{2!} + \binom{2}{1}u_{xt}\frac{xt}{2!} + \binom{2}{2}u_{t^2}\frac{t^2}{2!}$$

$$\frac{d^3u}{3!} = \binom{3}{0}u_{x^3}\frac{x^3}{3!} + \binom{3}{1}u_{x^2t}\frac{x^2t}{3!} + \binom{3}{2}u_{xt^2}\frac{xt^2}{3!} + \binom{3}{3}u_{t^3}\frac{t^3}{3!}$$

$$\frac{d^4u}{4!} = -\binom{4}{0}u_{t^2}\frac{x^4}{4!} + \binom{4}{1}u_{x^3t}\frac{x^3t}{4!} + \binom{4}{2}u_{x^2t^2}\frac{x^2t^2}{4!} + \binom{4}{3}u_{xt^3}\frac{xt^3}{4!} + \binom{4}{4}u_{t^4}\frac{t^4}{4!}$$

$$\frac{d^5u}{5!} = -\binom{5}{0}u_{t^2x}\frac{x^5}{5!} - \binom{5}{1}u_{t^3}\frac{x^4t}{5!} + \binom{5}{2}u_{x^3t^2}\frac{x^3t^2}{5!} + \binom{5}{3}u_{x^2t^3}\frac{x^2t^3}{5!} + \\ + \binom{5}{4}u_{xt^4}\frac{xt^4}{5!} + \binom{5}{5}u_{t^5}\frac{t^5}{5!}$$

...

Let us transform formula (2.3) with the use of (2.4), and then group the coefficients at consecutive derivatives. Finally, we obtain the following polynomials

$$S_{0,0} = 1$$

$$S_{1,0} = \binom{1}{0} x \qquad S_{1,1} = \binom{1}{1} t$$

$$S_{2,0} = \binom{2}{0}\frac{x^2}{2} \qquad S_{2,1} = \binom{2}{1}\frac{xt}{2}$$

$$S_{2,2} = \binom{2}{2}\frac{t^2}{2!} - \binom{4}{0}\frac{x^4}{4!}$$

$$S_{3,0} = \binom{3}{0}\frac{x^3}{3!} \qquad S_{3,1} = \binom{3}{1}\frac{x^2 t}{3!}$$

$$S_{3,2} = \binom{3}{2}\frac{xt^2}{3!} - \binom{5}{0}\frac{x^5}{5!} \qquad S_{3,3} = \binom{3}{3}\frac{t^3}{3!} - \binom{5}{1}\frac{x^4 t}{5!}$$

$$S_{4,1} = \binom{4}{1}\frac{x^3 t}{4!} \qquad S_{4,2} = \binom{4}{2}\frac{x^2 t^2}{4!} - \binom{6}{0}\frac{x^6}{6!}$$

$$S_{4,3} = \binom{4}{3}\frac{xt^3}{4!} - \binom{6}{1}\frac{x^5 t}{6!} \qquad S_{4,4} = \binom{4}{4}\frac{t^4}{4!} - \binom{6}{2}\frac{x^4 t^2}{6!} + \binom{8}{0}\frac{x^8}{8!}$$

$$\ldots$$

For $m = 0, 1, 2, \ldots$; $k = 0, 1, 2, 3$; $m - k \geqslant 0$ the polynomials have a form

$$S_{m,m-k} = \sum_{j=0}^{[\frac{m-k}{2}]} \frac{(-1)^j t^{m-k-2j} x^{k+4j}}{(m-k-2j)!(k+4j)!} \tag{2.5}$$

where $[x] = floor(x)$. It is convenient to renumber polynomials (2.5) according to Table 1.

Table 1. New numbering of the polynomials

$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	...
0,0 **0**	1,0 **1**	2,0 **3**	3,0 **6**	4,1 **10**	5,2 **14**	6,3 **18**	...
	1,1 **2**	2,1 **4**	3,1 **7**	4,2 **11**	5,3 **15**	6,4 **19**	...
		2,2 **5**	3,2 **8**	4,3 **12**	5,4 **16**	6,5 **20**	...
			3,3 **9**	4,4 **13**	5,5 **17**	6,6 **21**	...

Then we get

$$
\begin{aligned}
S_0 &= S_{0,0}(x,t) = 1 & S_1 &= S_{1,0}(x,t) = x \\
S_2 &= S_{1,1}(x,t) = t & S_3 &= S_{2,0}(x,t) = \frac{x^2}{2} \\
S_4 &= S_{2,1}(x,t) = xt & S_5 &= S_{2,2}(x,t) = \frac{t^2}{2!} - \frac{x^4}{4!}
\end{aligned}
$$

and for $m \geqslant 3$ and $k = 0, 1, 2, 3$

$$S_n = S_{4m-k-3} = S_{m,m-k} = \sum_{j=0}^{[\frac{m-k}{2}]} \frac{(-1)^j t^{m-k-2j} x^{k+4j}}{(m-k-2j)!(k+4j)!}$$

Let us denote $k = 4[\frac{n-2}{4}] - n + 5$ and $m = [\frac{n-2}{4}] + 2$. Then, for $n \geqslant 3$, we obtain

$$S_n(x,t) = \sum_{j=0}^{[\frac{n-3[\frac{n-2}{4}]-3}{2}]} \frac{(-1)^j t^{n-3[\frac{n-2}{4}]-3-2j} x^{(4[\frac{n-2}{4}]-n+5)+4j}}{(n-3[\frac{n-2}{4}]-3-2j)!((4[\frac{n-2}{4}]-n+5)+4j)!} \tag{2.6}$$

2.2. Partial derivatives of th solving polynomials

In numerical practice, recurrent formulas are very useful. Theorem 1 presents such formulas for the derivatives of solving polynomials in the beam vibration problem. Properties shown in the theorem prove that the considered polynomials satisfy equation (2.2). That is why they are called the "solving polynomials".

Theorem 1

For polynomials S_n, we get:
a) for $n \geqslant 5$

$$\frac{\partial S_n}{\partial x} = \begin{cases} S_{n-3} & \text{for} \quad n \neq 4p+1, \ p = 1, 2, \dots \\ -S_{n+1} & \text{for} \quad n = 4p+1, \ p = 1, 2, \dots \end{cases} \tag{2.7}$$

b) for $n \geqslant 8$

$$\frac{\partial^2 S_n}{\partial x^2} = \begin{cases} S_{n-6} & \text{for} \quad n = 4p+2 \ \text{ or } \ n = 4p+3, \ p = 2, 3, \dots \\ -S_{n-2} & \text{for} \quad n = 4p \ \text{ or } n = 4p+1, \ p = 2, 3, \dots \end{cases} \tag{2.8}$$

c) for $n \geqslant 11$

$$\frac{\partial^3 S_n}{\partial x^3} = \begin{cases} S_{n-9} & \text{for} \quad n = 4p+2, \;\; p = 3, 4, \ldots \\ -S_{n-5} & \text{for} \quad n \neq 4p+2, \;\; p = 3, 4, \ldots \end{cases} \tag{2.9}$$

d) for $n \geqslant 11$

$$\frac{\partial^4 S_n}{\partial x^4} = -S_{n-8} \tag{2.10}$$

e) for $n \geqslant 7$

$$\frac{\partial S_n}{\partial t} = S_{n-4} \tag{2.11}$$

f) for $n \geqslant 11$

$$\frac{\partial^2 S_n}{\partial t^2} = S_{n-8} \tag{2.12}$$

Properties (2.10) and (2.12) prove that the polynomials S_n satisfy equation (2.2).

Proof of the Theorem 1
a) For $n \geqslant 3$

$$S_n(x,t) = \sum_{j=0}^{\left[\frac{n-3[\frac{n-2}{4}]-3}{2}\right]} \frac{(-1)^j t^{n-3[\frac{n-2}{4}]-3-2j} x^{(4[\frac{n-2}{4}]-n+5)+4j}}{(n-3[\frac{n-2}{4}]-3-2j)!((4[\frac{n-2}{4}]-n+5)+4j)!}$$

In the first summand of S_n, the variable x to the power of $w = 4[\frac{n-2}{4}] - n + 5$ appears. It can be easily shown that for $n = 4p$; $n = 4p+2$; $n = 4p+3$ $(p = 1, 2, \ldots)$ we obtain $w \geqslant 1$, and for $n = 4p+1$ we get $w = 0$. Then, for $n = 4p$

$$\frac{\partial S_n(x,t)}{\partial x} = \sum_{j=0}^{\left[\frac{n-3[\frac{n-2}{4}]-3}{2}\right]} \frac{(-1)^j t^{n-3[\frac{n-2}{4}]-3-2j} x^{(4[\frac{n-2}{4}]-n+4)+4j}}{(n-3[\frac{n-2}{4}]-3-2j)!((4[\frac{n-2}{4}]-n+4)+4j)!} =$$

$$= \sum_{j=0}^{\left[\frac{4p-3[\frac{4p-2}{4}]-3}{2}\right]} \frac{(-1)^j t^{4p-3[\frac{4p-2}{4}]-3-2j} x^{(4[\frac{4p-2}{4}]-4p+4)+4j}}{(4p-3[\frac{4p-2}{4}]-3-2j)!((4[\frac{4p-2}{4}]-4p+4)+4j)!} =$$

$$= \sum_{j=0}^{[\frac{p}{2}]} \frac{(-1)^j t^{p-2j} x^{4j}}{(p-2j)!(4j)!}$$

On the other hand,

$$S_{n-3}(x,t) = S_{4p-3}(x,t) =$$

$$= \sum_{j=0}^{\left[\frac{4p-3-3[\frac{4p-3-2}{4}]-3}{2}\right]} \frac{(-1)^j t^{4p-3-3[\frac{4p-3-2}{4}]-3-2j} x^{(4[\frac{4p-3-2}{4}]-4p-3+4)+4j}}{(4p-3-3[\frac{4p-3-2}{4}]-3-2j)!((4[\frac{4p-3-2}{4}]-(4p-3)+4)+4j)!} =$$

$$= \sum_{j=0}^{[\frac{p}{2}]} \frac{(-1)^j t^{p-2j} x^{4j}}{(p-2j)!(4j)!}$$

This proves property a) for $n = 4p$. For $n = 4p+2$; $n = 4p+3$ the proof is similar.
For $n = 4p+1$ we get

$$\frac{\partial S_n(x,t)}{\partial x} =$$

$$= \sum_{j=1}^{\left[\frac{4p+1-3[\frac{4p+1-2}{4}]-3}{2}\right]} \frac{(-1)^j t^{4p+1-3[\frac{4p+1-2}{4}]-3-2j} x^{(4[\frac{4p+1-2}{4}]-(4p+1)+4)+4j}}{(4p+1-3[\frac{4p+1-2}{4}]-3-2j)!((4[\frac{4p+1-2}{4}]-(4p+1)+4)+4j)!} =$$

$$= \sum_{j=1}^{[\frac{p+1}{2}]} \frac{(-1)^j t^{p+1-2j} x^{-1+4j}}{(p+1-2j)!(-1+4j)!} =$$

substituting $j = s+1$

$$= -\sum_{s=0}^{[\frac{p-1}{2}]} \frac{(-1)^s t^{p-1-2s} x^{3+4s}}{(p-1-2s)!(3+4s)!}$$

On the other hand, for $n = 4p+1$

$$-S_{n+1}(x,t) =$$

$$= -\sum_{j=0}^{\left[\frac{4p+2-3[\frac{4p}{4}]-3}{2}\right]} \frac{(-1)^j t^{4p+2-3[\frac{4p}{4}]-3-2j} x^{(4[\frac{4p}{4}]-(4p+2)+5)+4j}}{(4p+2-3[\frac{4p}{4}]-3-2j)!((4[\frac{4p}{4}]-(4p+2)+5)+4j)!} =$$

$$= -\sum_{j=0}^{[\frac{p-1}{2}]} \frac{(-1)^j t^{p-1-2j} x^{3+4j}}{(p-1-2j)!(3+4j)!}$$

This completes the proof of property a). Properties b), c), d) follow property a).

e) We have

$$S_n(x,t) = \sum_{j=0}^{\left[\frac{n-3[\frac{n-2}{4}]-3}{2}\right]} \frac{(-1)^j t^{n-3[\frac{n-2}{4}]-3-2j} x^{(4[\frac{n-2}{4}]-n+5)+4j}}{(n-3[\frac{n-2}{4}]-3-2j)!((4[\frac{n-2}{4}]-n+5)+4j)!}$$

The variable t appears in the derivative $\partial S_n/\partial t$ if and only if in S_n the inequality $w = n-3[\frac{n-2}{4}]-3-2j \geqslant 1$ holds. Hence, for j we get $j \leqslant (n-3[\frac{n-2}{4}]-4)/2$ and

$$\frac{\partial S_n(x,t)}{\partial t} = \sum_{j=0}^{\left[\frac{n-3[\frac{n-2}{4}]-4}{2}\right]} \frac{(-1)^j t^{n-3[\frac{n-2}{4}]-4-2j} x^{(4[\frac{n-2}{4}]-n+5)+4j}}{(n-3[\frac{n-2}{4}]-4-2j)!((4[\frac{n-2}{4}]-n+5)+4j)!}$$

On the other hand

$$S_{n-4} = \sum_{j=0}^{\left[\frac{n-3[\frac{n-2}{4}-1]-7}{2}\right]} \frac{(-1)^j t^{n-3[\frac{n-2}{4}-1]-7-2j} x^{(4[\frac{n-2}{4}-1]-n+9)+4j}}{(n-3[\frac{n-2}{4}-1]-7-2j)!((4[\frac{n-2}{4}-1]-n+9)+4j)!} =$$

$$= \sum_{j=0}^{\left[\frac{n-3[\frac{n-2}{4}]-4}{2}\right]} \frac{(-1)^j t^{n-3[\frac{n-2}{4}]-4-2j} x^{(4[\frac{n-2}{4}]-n+5)+4j}}{(n-3[\frac{n-2}{4}]-4-2j)!((4[\frac{n-2}{4}]-n+5)+4j)!}$$

This proves property e). Property f) results directly from e).

2.3. Generating function

Using the method of variable separation in equation (2.2), we arrive at the function $W(x,t,p)$ called the generating function for the solving functions

$$W(x,t,p) = \mathrm{e}^{px+\mathrm{i}p^2t} \tag{2.13}$$

with $\mathrm{i} = \sqrt{-1}$. Expanding the function W into the power series, we obtain

$$W(x,t,p) = \mathrm{e}^{px+\mathrm{i}p^2t} =$$
$$= 1 + px + \frac{p^2}{2!}(2\mathrm{i}t + x^2) + \frac{p^3}{3!}(6\mathrm{i}tx + x^3) + \frac{p^4}{4!}(-12t^2 + 12\mathrm{i}tx^2 + x^4) + \ldots =$$
$$= S_0 + pS_1 + p^2(\mathrm{i}S_2 + S_3) + p^3(-\mathrm{i}S_4 + S_6) + p^4(-S_5 + \mathrm{i}S_7) + \ldots$$

It is clear that the real and imaginary parts of coefficients at successive powers of the parameter p are the previously obtained solving polynomials.

3. The method of solving functions

The method of solving functions discussed here belongs to a class of the Trefftz methods. The solving polynomials may be used to obtain an approximate solution to equation (2.2) with prescribed initial and boundary conditions. We can determine a solution in the whole domain $(0,1)\times(0,\Delta t)$ or we can use the polynomials as base functions in the Finite Element Method. In the second case, at least three approaches are applied: continuous FEM, non-continuous FEM (Ciałkowski and Frąckowiak, 2007) or nodeless FEM (Maciąg, 2007b). Unfortunately, for beam vibration problems, the use of the solving polynomials in continuous and discontinuous FEM does not lead to satisfactory results. The matrix obtained in FEM is then ill-conditioned. This problem does not appear in the nodeless FEM. Therefore, in this paper we consider two cases: how to use the solving polynomials in the whole domain $(0,1)\times(0,\Delta t)$ and how to use them in the nodeless FEM. In both cases we take a linear combination of the polynomials in order to find an approximate solution (in the whole domain or in time-space elements)

$$u \approx w = \sum_{n=1}^{N} c_n S_n \tag{3.1}$$

Because the polynomials S_n satisfy equation (2.2), their linear combination satisfies this equation as well. The coefficients c_n are chosen to obtain the best fitting to the initial and boundary conditions. Additionally, in the nodeless FEM we have to minimize the difference of solutions between the elements (see example).

4. Example

The goal of the example presented here is not to show all possible cases of beam vibration. The authors want only to show how to find an approximate solution to the beam vibration problem with satisfactory accuracy using the method of solving functions. In numerical practice, the "exact (analytical) solution" very often does not lead to accurate numerical values of solution because of truncation errors. For vibrations of a beam, an analytical solution includes roots of a transcendental equation. That may cause some numerical problems.

The method is tested here for a problem for which the exact solution is known. We consider two ways of using the solving polynomials. The first one is the use of the polynomials in the whole domain. In the second approach, the solving functions stands for base functions in the nodeless Finite Element Method.

4.1. Problem formulation

Consider vibrations of a beam which is fixed at $x=0$ and free at $x=1$:
— equation of motion

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad x \in (0,1), \quad t > 0 \tag{4.1}$$

— initial conditions

$$u(x,0) = u_0(x) = \frac{1}{1000} x^2 \tag{4.2}$$

$$\frac{\partial u(x,0)}{\partial t} = 0 \tag{4.3}$$

— boundary conditions, $t > 0$

$$u(0,t) = 0 \tag{4.4}$$

$$\frac{\partial u(0,t)}{\partial x} = 0 \tag{4.5}$$

$$\frac{\partial^2 u(x,1)}{\partial x^2} = 0 \tag{4.6}$$

$$\frac{\partial^3 u(x,1)}{\partial x^3} = 0 \tag{4.7}$$

The exact solution to the problem reads

$$u(x,t) = \sum_{n=1}^{\infty} \frac{K_n(x)\cos(\gamma_n^2 t)\int_0^1 K_n(x) u_0(x)\, dx}{(\sin\gamma_n + \sinh\gamma_n)^2} \tag{4.8}$$

where

$$\begin{aligned} K_n(x) = (\cos\gamma_n + \cosh\gamma_n)[\sinh(\gamma_n x) - \sin(\gamma_n x)] + \\ -(\sin\gamma_n + \sinh\gamma_n)[\cosh(\gamma_n x) - \cos(\gamma_n x)] \end{aligned}$$

and γ_n are successive positive roots of the equation $1 + \cos x \cosh x = 0$.

4.2. Approximate solution in the whole domain

An approximate solution $u(x,t)$ has a form as follows

$$u \approx w = \sum_{n=1}^{N} c_n S_n$$

Because the function w satisfies the equation of motion of the vibrating beam (with w being a linear combination of the solving polynomials), in order to find the coefficients c_n we minimize the norm $\|w-u\|$ for the boundary and initial conditions. We look for an approximate solution u in the time interval $(0, \Delta t)$. Hence, the coefficients c_n have to be appropriately chosen to minimize the functional

$$\begin{aligned} I = & \underbrace{\int_0^1 [w(x,0) - u_0(x)]^2 \, dx}_{\text{cond. (4.2)}} + \underbrace{\int_0^1 \Big[\frac{\partial w(x,0)}{\partial t} - 0\Big]^2 \, dx}_{\text{cond. (4.3)}} \\ & + \underbrace{\int_0^{\Delta t} [w(0,t) - 0]^2 \, dt}_{\text{cond. (4.4)}} + \underbrace{\int_0^{\Delta t} \Big[\frac{\partial w(0,t)}{\partial x} - 0\Big]^2 \, dt}_{\text{cond. (4.5)}} \\ & + \underbrace{\int_0^{\Delta t} \Big[\frac{\partial^2 w(1,t)}{\partial x^2} - 0\Big]^2 \, dt}_{\text{cond. (4.6)}} + \underbrace{\int_0^{\Delta t} \Big[\frac{\partial^3 w(1,t)}{\partial x^3} - 0\Big]^2 \, dt}_{\text{cond. (4.7)}} \end{aligned} \tag{4.9}$$

The necessary condition to minimize the functional I is

$$\frac{\partial I}{\partial c_1} = \ldots = \frac{\partial I}{\partial c_N} = 0 \tag{4.10}$$

The linear system of equations (4.10) can be written as

$$\mathbf{AC} = \mathbf{B} \tag{4.11}$$

where $\mathbf{C} = [c_1, \ldots, c_N]^\top$ and the elements of matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$
\begin{aligned}
a_{ij} = & \int_0^1 S_i(x,0) S_j(x,0)\, dx + \int_0^1 \frac{\partial S_i(x,0)}{\partial t} \frac{\partial S_j(x,0)}{\partial t}\, dx \\
& + \int_0^{\Delta t} S_i(0,t) S_j(0,t)\, dt + \int_0^{\Delta t} \frac{\partial S_i(0,t)}{\partial x} \frac{\partial S_j(0,t)}{\partial x}\, dt \\
& + \int_0^{\Delta t} \frac{\partial^2 S_i(1,t)}{\partial x^2} \frac{\partial^2 S_j(1,t)}{\partial x^2}\, dt + \int_0^{\Delta t} \frac{\partial^3 S_i(1,t)}{\partial x^3} \frac{\partial^3 S_j(1,t)}{\partial x^3}\, dt \\
b_i = & \int_0^1 S_i(x,0) u_0(x)\, dx
\end{aligned}
$$

Note that $a_{ij} = a_{ji}$ (the matrix $\mathbf{A}$ is symmetric) which simplifies the calculations. From equation (4.11), we obtain the coefficients c_n: $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$.

In the time intervals $(\Delta t, 2\Delta t)$, $(2\Delta t, 3\Delta t)$, ..., we proceed analogously. Here, the initial condition for the time interval $((m-1)\Delta t, m\Delta t)$ is the value of the function u at the end of the interval $((m-2)\Delta t, (m-1)\Delta t)$. All results shown in Figures 1-4 have been obtained for $\Delta t = 1$. The number N of polynomials in formula (3.1) depends on the degree D of the polynomials. Here $N = 2D + 1$.

In the whole time interval $(0, \Delta t)$ a good approximation is obtained. For example, Fig. 1 shows (a) the exact solution, (b) the approximation with the solving polynomials of degrees up to 30, (c) the difference between (a) and (b). It is clear that the approximation is satisfactory. The largest error has been obtained for $x = 1$ (for $x = 0$ the beam is fixed).

Figure 2 shows the initial condition $u(x, 0)$ in the case of (a) the exact solution, (b) the approximation with the solving polynomials of degrees up 30, (c) the difference.

Figure 3 shows the solution $u(x, 0.5)$ in the case of (a) the exact solution, (b) the approximation with the solving polynomials of degrees up to 30, (c) the difference.

Figures 2 and 3 show that the approximation with polynomials of degrees up to 30 is satisfactory. The determination of the highest degree of the solving polynomials used in approximation (3.1) is very important. Our goal is to minimize the inaccuracy of the approximation. The solution should be "as accurate as possible". The method of solving functions should be convergent. It means that using more polynomials one should obtain a better approximation.

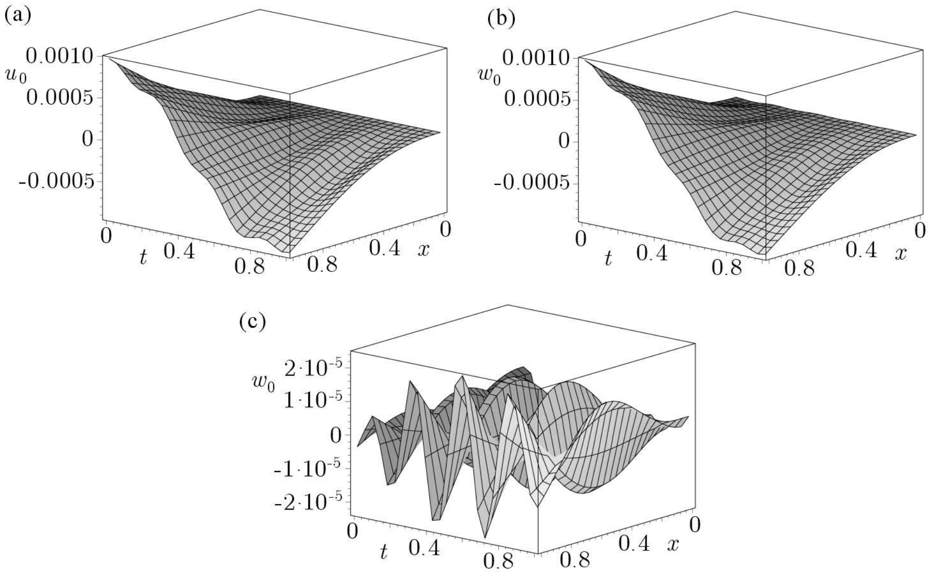

Fig. 1. Solutions (a) exact, (b) approximated with the solving polynomials of degrees up 30, (c) difference

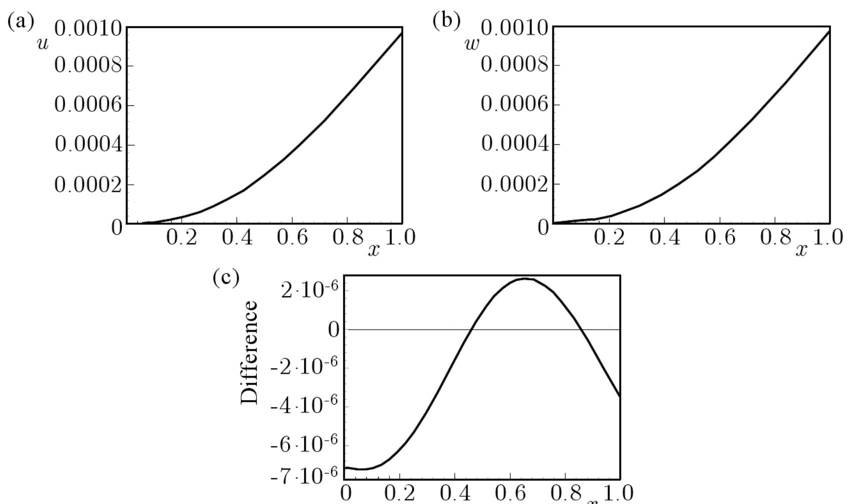

Fig. 2. The initial condition in the case of (a) exact solution, (b) approximation with the solving polynomials of degrees up to 30, (c) difference

In general, it is true. Unfortunately, a too big number of polynomials leads to big dimensions of the matrix in (4.11). That may cause numerical problems. Figure 4 shows the exact and approximate solution for $x = 1$ with the solving polynomials of degrees up to (a) 10, (b) 25, (c) 30.

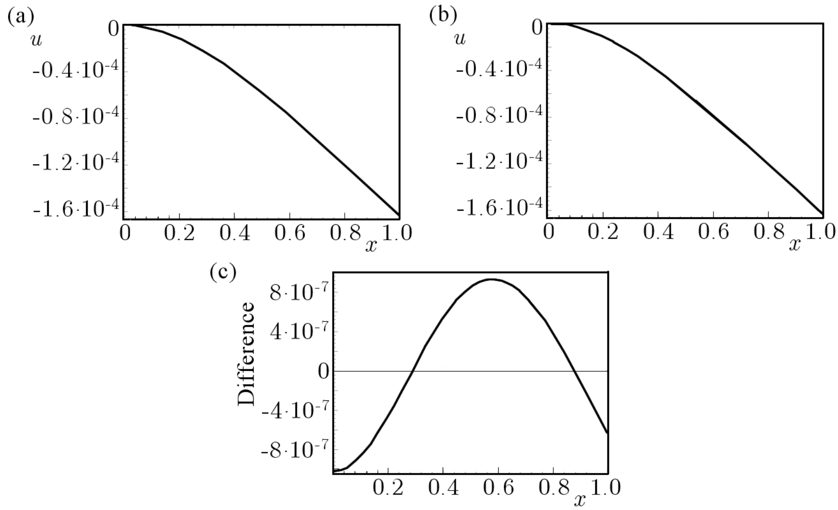

Fig. 3. Solution for $t = 0.5$ in the case of (a) exact solution, (b) approximation with the solving polynomials of degrees up to 30, (c) difference

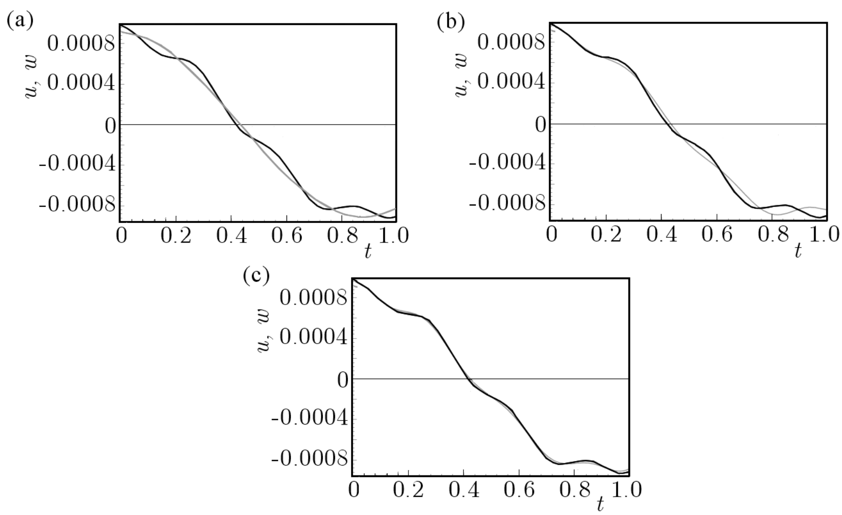

Fig. 4. Exact and approximate solutions for $x = 1$ with the solving polynomials of degrees up to (a) 10, (b) 25, (c) 30

Let $w_D(1,t)$ denote the approximation $w(1,t)$ with polynomials of degrees 0 to D. The average, relative difference between the solutions $w_D(1,t)$ and $u(1,t)$ may be defined as follows

$$E(D) = \sqrt{\frac{\int_0^{\Delta t} [w_D(1,t) - u(1,t)]^2 \, dt}{\int_0^{\Delta t} [u(1,t)]^2 \, dt}}$$

Table 2 shows the difference $E(D)$ which depends on the highest degree D of the solving polynomial.

Table 2. Error in relation to the highest degree D of the solving polynomial

Order D	10	15	20	25	30
$E(D)$	0.0917	0.0884	0.0843	0.0741	0.0221

The difference $E(D)$ decreases when the number of the polynomials in the approximation u increases. It means that the truncation error for the considered values of D is negligible, and that the procedure is convergent.

4.3. Nodeless Finite Element Method

The second way of using solving polynomials is the nodeless FEM. Let us divide the interval $(0,1)$ into subintervals and seek the solution in the time-space subdomains $L_k = ((k-1)\Delta x, k\Delta x) \times \Delta t$, $k = 1, \dots, K$. Let us introduce a local co-ordinate system in each subdomain L_k and assume the approximation of the solution in the form

$$u_k \approx w_k = \sum_{n=1}^{N} c_n^k S_n \tag{4.12}$$

To find the coefficients c_n^k, we minimize the norm $\|w - u\|$ for the boundary and initial conditions. Moreover, the norms: $\|w_{k-1} - w_k\|$, $\|\frac{\partial w_{k-1}}{\partial x} - \frac{\partial w_k}{\partial x}\|$, $\|\frac{\partial^2 w_{k-1}}{\partial x^2} - \frac{\partial^2 w_k}{\partial x^2}\|$ and $\|\frac{\partial^3 w_{k-1}}{\partial x^3} - \frac{\partial^3 w_k}{\partial x^3}\|$ are minimized simultaneously. In the time intervals $(\Delta t, 2\Delta t)$, $(2\Delta t, 3\Delta t)$, $\dots$, we proceed analogously. Here, the value of the function w at the end of the interval $((m-2)\Delta t, (m-1)\Delta t)$ stands for the initial condition for the time interval $((m-1)\Delta t, m\Delta t)$. In order to compare the solution in the whole interval $x \in (0,1)$ with the approximate solution obtained with the nodeless FEM we assume $\Delta t = 1/2$. Figure 5 shows the exact solution, approximation with the solving polynomials of degrees up to 20 and the difference for $x = 1$, for the number of subintervals $K = 2$ and for the number of time steps‘ $TS = 2$.

When comparing Fig. 4 and Fig. 5, we notice that for the nodeless FEM the approximation is better than for the solution in the whole domain. Moreover,

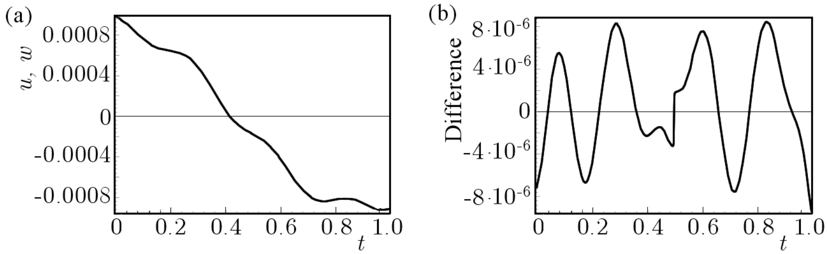

Fig. 5. (a) Exact and approximate solutions for $x = 1$ (b) the difference

using only two subintervals (i.e. K=2) allowed one to reduce the highest degree of the solving polynomials from 30 down to 20. When using the nodeless FEM, the advantage is such that the elements can be relatively big. Moreover, in each element we have a solution satisfying the equation.

Like in Section 4.2, we define the relative error of the approximate solution, $E(D)$, as follows

$$E(D) = \sqrt{\frac{\int_0^{TS\Delta t} [w_D(1,t) - u(1,t)]^2 \, dt}{\int_0^{TS\Delta t} [u(1,t)]^2 \, dt}} \tag{4.13}$$

where TS stands for the number of time steps. Table 3 shows values of the error $E(D)$ for $TS = 2$, $\Delta t = 1/2$, which depend on the highest degree, D, and the number of subintervals, K.

Table 3. Error dependence on the polynomial highest degree, D, and number of subintervals K

	$K = 2$	$K = 3$	$K = 4$	$K = 5$
$D = 10$	0.0928	0.0951	0.0943	0.0932
$D = 15$	0.06	0.0515	0.0436	0.0433
$D = 20$	0.0073	0.011	0.0092	0.008

It is clear that the number of polynomials used in the approximation has the greatest influence on the error, $E(D)$. Generally, the increasing number of subintervals decreases the error, but the relationship is not so distinct: probably the truncation errors increase in that case. When comparing Table 2 and 3, we see that the nodeless FEM leads to a more accurate approximation than the solution in the whole domain.

It is interesting how the nodeless FEM works for a greater number of time steps; the method should then give "satisfactory" results. However, when we increase the number of time steps, the error of approximation increases too because of the truncation errors. And because the solutions for each time step are not accurate, the error propagates in time. Figure 6 shows the exact solution, approximation with the solving polynomials of degrees up to 20 and the difference for $x = 1$ with the number of subintervals $K = 2$ and the number of time steps $TS = 10$.

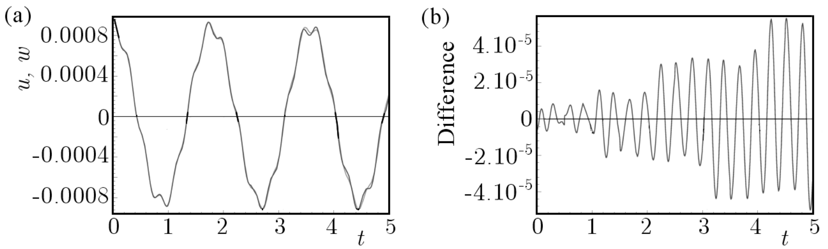

Fig. 6. (a) Exact and approximate solutions for $x = 1$ (b) the difference

Table 4 shows $E(D)$ for $\Delta t = 1/2$, degree $D = 20$ and the number of subintervals $K = 2$ as a function of the number of time steps TS.

Table 4. Error in relation to the number of time steps

TS	2	4	6	8	10
$E(20)$	0.0073	0.013	0.02	0.028	0.035

The error increases when the number of time steps increases too, but remains small. For 10 time steps, the error does not exceed 3.5%, which shows that the approximation is good.

5. Concluding remarks

A new simple technique for solving the problem of beam vibration has been developed. The test example shows that the obtained approximations are very good. The solution, which is a linear combination of the solving polynomials, exactly satisfies the differential equation of motion and – approximately – the initial and boundary conditions.

The solving polynomials presented here lead to good results both in the whole domain and in the nodeless FEM. In the second case, the elements can be relatively big. The main advantage of the method of solving functions is that the approximate solution of the considered problem satisfies the governing equation.

Acknowledgments

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Metoda wielomianów rozwiązujących dla problemu drgań belki

Streszczenie

Artykuł przedstawia nową metodę przybliżonego rozwiązywania problemów drgań belki, które opisywane są cząstkowym równaniem różniczkowym czwartego rzędu. Generalnie takie równania mogą być rozwiązywane różnymi metodami. Główna idea prezentowanego tutaj podejścia polega na znalezieniu wielomianów spełniających w sposób ścisły dane równanie różniczkowe (funkcje rozwiązujące). Za rozwiązanie przybliżone przyjmuje się kombinację liniową tych funkcji, która również spełnia równanie. Współczynniki kombinacji liniowej wyznaczane są tak, aby uzyskane rozwiązanie w sposób najlepszy (w sensie średniokwadratowym) było dopasowane do danych warunków początkowych i brzegowych. Jest to zatem wariant metody Trefftza.

W pracy wyznaczono efektywne wzory dla funkcji rozwiązujących i ich pochodnych. Uzyskane wielomiany mogą być wykorzystane do wyznaczenia rozwiązania w całym obszarze lub też mogą być użyte jako funkcje bazowe w bezwęzłowej Metodzie Elementów Skończonych. Oba przypadki rozważono w pracy. Załączono również przykłady numeryczne.

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