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SINGULARITY OF STRESSES IN A PERIODIC LAMINATED SEMI-SPACE WITH A BOUNDARY NORMAL TO THE LAYERING

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The paper deals with the plane problem of stress distributions in a non-homogeneous elastic semi-space caused by concentrated loadings acting on its boundary. The body is composed of periodically repeated two-constituent laminae, and its boundary is assumed to be normal to the layering. The solution to the problem is presented within the framework of the homogenized model with microlocal parameters (Woźniak, 1987; Matysiak and Woźniak, 1987). Analytical results for stresses are obtained and shown in figures.

Key words: composite, displacement, stresses, elasticity, homogenized model, concentrated loading

1. Introduction

Layered elastic materials with periodic structures represent an important type of composites, which can be found in nature (varved clays, sandstone-slates, sandstone-shales, thin-layered limestones) or are made by man and used in various engineering technologies. The behavior of systems, which are made of a large number of repeated laminae, is described within the framework of elasticity theory by partial differential equations with discontinuous, oscillating coefficients. Thus formulated problems are too complicated for analytical and numerical approaches. For this reason, applications of some approximate models seem to be useful. One of the models is the homogenized model with microlocal parameters devised by Woźniak (1987) and developed for microperiodic layered elastic composites by Matysiak and Woźniak (1987).

This model satisfies continuity conditions for displacements and stress vectors on interfaces. The homogenized model with microlocal parameters was applied to solve many problems, which were partially reassumed in papers of Matysiak (1995), Woźniak and Woźniak (1995). The two-dimensional problem of stress distribution in a layered semi-space caused by a concentrated loading acting on the boundary was solved by Kaczyński and Matysiak (1987) in the case of a boundary plane parallel to the layering. The problem of concentrated moving loads on the boundary with steady supersonic velocity for the laminated semi-space was discussed within the framework of the "effective stiffness theory" by Sve and Hermann (1974).

In this paper, a periodically two-layered elastic semi-space in the plane state of strain is considered. The boundary plane is normal to the layering. The concentrated loading with the intensity σ_0 acting at an arbitrary point of the boundary and inclined with the angle θ to the boundary is considered. The problem is solved within the framework of the homogenized model, which satisfies continuity conditions on the interfaces. The component of normal stresses to the boundary is discontinuous on the interfaces, what leads to some discontinuous, oscillating boundary condition. In this paper, that condition is approximated by averaged ones. Thus, the stress distributions in the laminated elastic semi-space are determined and presented in form of graphs.

The results obtained within the framework of the homogenized model with microlocal parameters for certain heat condition problems were compared with the adequate solutions determined by the classical description by Kulchytsky-Zhyailo and Matysiak (2006). Moreover, the analogous analysis of boundary value problems of periodically layered elastic composites was derived by Kulchytsky-Zhyhailo *et al.* (2006). In both papers, a good agreement of the solutions obtained within the framework of the homogenized model and the classical approach is observed.

2. Formulation of the problem

Consider a two-dimensional static problem of a half-space with the layering normal to the boundary plane, (see Fig. 1). Let λ_i, μ_i, $i = 1,2$ be Lamés constants and l_i, $i = 1,2$ be the thicknesses of subsequent layers being the composite constituents, and l, $l = l_1 + l_2$ be the thickness of fundamental lamina. The perfect bonding between the layers is assumed. Referring to the Cartesian coordinate system (x, y) with the y – axis directed parallel to the layering and the x – axis being one of the interfaces, denote at the point (x, y) the displacement vector $[u(x, y), v(x, y), 0]$ and the stress component $\sigma_{xx}^{(j)}$, $\sigma_{xy}^{(j)}$, $\sigma_{yy}^{(j)}$, $\sigma_{zz}^{(j)}$, $j = 1,2$, in the j-th kind of composite constituent. Let the nonhomogeneous half-space be loaded at the point $(a, 0)$ by a concentrated force with the intensity σ_0 inclined with the angle θ to the x-axis, see Fig. 1.

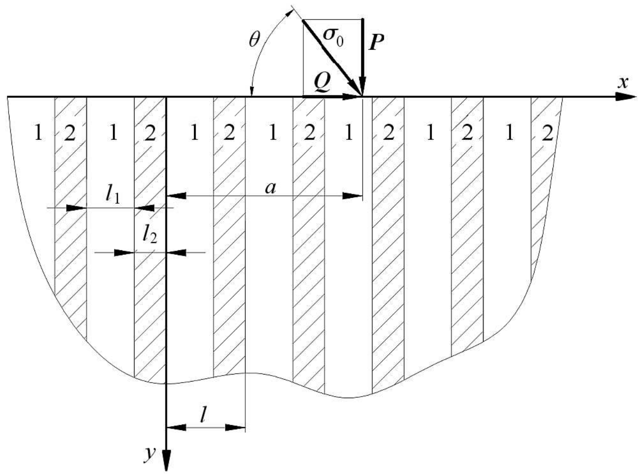

Fig. 1. A cross-section of the periodically laminated half-space

To solve this problem, the homogenized model with microlocal parameters will be applied. This model, presented in the general case of periodic thermoelastic composites by Woźniak (1987) and adopted to layered structures by Matysiak and Woźniak (1987), was employed in many problems of periodically stratified elastic and thermoelastic bodies. We recall only a brief outline of the governing equations in the case of plane state of strains.

The displacement vector is postulated in the form

$$
\begin{aligned}
u(x,y) &= U(x,y) + \underline{h(x)q_x(x,y)} \\
v(x,y) &= V(x,y) + \underline{h(x)q_y(x,y)}
\end{aligned}
\tag{2.1}
$$

where U, V are unknown functions interpreted as macrodisplacements, and q_x, q_y stand for microlocal parameters and are related to the periodic structure of the body. The function h is known *a priori* l-periodic function given in the form (see for example, Kaczyński and Matysiak, 1987)

$$
\begin{aligned}
&h(x) = \begin{cases} x - \dfrac{1}{2}l_1 & \text{for} \quad 0 \leqslant x \leqslant l_1 \\[2mm] \dfrac{-\eta x}{1-\eta} - \dfrac{1}{2}l_1 + \dfrac{l_1}{1-\eta} & \text{for} \quad l_1 \leqslant x \leqslant l \end{cases} \\
&h(x+l) = h(x) \qquad\qquad \eta = \frac{l_1}{l}
\end{aligned}
\tag{2.2}
$$

Since for every x $|h(x)| < l$, then for small l the underlined terms in (2.1) are small too and will be neglected. The derivative $h'(x)$ is not small however, so it determines the following approximations

$$
\begin{array}{lll}
u \approx U & v \approx V & \dfrac{\partial u}{\partial x} \approx \dfrac{\partial U}{\partial x} + h_j q_x \\
\dfrac{\partial v}{\partial x} \approx \dfrac{\partial V}{\partial x} + h_j q_y & \dfrac{\partial u}{\partial y} \approx \dfrac{\partial U}{\partial y} & \dfrac{\partial v}{\partial y} \approx \dfrac{\partial V}{\partial y}
\end{array}
\tag{2.3}
$$

where h_j, $j = 1,2$ are derivatives of the function $h(x)$ in the jth kind of composite component

$$
h_1 = 1 \qquad\qquad h_2 = -\frac{\eta}{1-\eta} \tag{2.4}
$$

The homogenized model with microlocal parameters in the plane state of strains is described by the following equations (see Kaczyński and Matysiak, 1987)

$$
\begin{aligned}
A_1 \frac{\partial^2 U}{\partial x^2} + C \frac{\partial^2 U}{\partial y^2} + (B+C) \frac{\partial^2 V}{\partial x \partial y} &= 0 \\
C \frac{\partial^2 V}{\partial x^2} + A_2 \frac{\partial^2 V}{\partial y^2} + (B+C) \frac{\partial^2 U}{\partial x \partial y} &= 0
\end{aligned}
\tag{2.5}
$$

and the stresses in the jth kind of composite constituent $(j = 1,2)$

$$
\begin{array}{ll}
\sigma_{xy}^{(j)} = C\Big(\dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial x}\Big) & \sigma_{xx}^{(j)} = A_1 \dfrac{\partial U}{\partial x} + B \dfrac{\partial V}{\partial y} \\
\sigma_{yy}^{(j)} = D_j \dfrac{\partial U}{\partial x} + E_j \dfrac{\partial V}{\partial y} & \sigma_{zz}^{(j)} = \dfrac{\lambda_j}{\lambda_j + 2\mu_j}(\sigma_{xx}^{(j)} + \sigma_{yy}^{(j)})
\end{array}
\tag{2.6}
$$

where

$$
\begin{aligned}
A_1 &= \widetilde{\lambda} + 2\widetilde{\mu} - \frac{([\lambda]+2[\mu])^2}{\widehat{\lambda} + 2\widehat{\mu}} = \frac{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}{(1-\eta)(\lambda_1 + 2\mu_1) + \eta(\lambda_2 + 2\mu_2)} > 0 \\
A_2 &= \widetilde{\lambda} + 2\widetilde{\mu} - \frac{[\lambda]^2}{\widehat{\lambda} + 2\widehat{\mu}} = A_1 + \frac{4\eta(1-\eta)(\mu_1 - \mu_2)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)}{(1-\eta)(\lambda_1 + 2\mu_1) + \eta(\lambda_2 + 2\mu_2)} > 0 \\
B &= \widetilde{\lambda} - \frac{[\lambda]([\lambda]+2[\mu])}{\widehat{\lambda} + 2\widehat{\mu}} = \frac{(1-\eta)\lambda_2(\lambda_1 + 2\mu_1) + \eta\lambda_1(\lambda_2 + 2\mu_2)}{(1-\eta)(\lambda_1 + 2\mu_1) + \eta(\lambda_2 + 2\mu_2)} > 0 \\
C &= \widetilde{\mu} - \frac{[\mu]^2}{\widehat{\mu}} = \frac{\mu_1 \mu_2}{(1-\eta)\mu_1 + \eta\mu_2} > 0 \qquad\qquad D_j = \frac{\lambda_j}{\lambda_j + 2\mu_j} A_1 \\
E_j &= \frac{4\mu_j(\lambda_j + \mu_j)}{\lambda_j + 2\mu_j} + \frac{\lambda_j}{\lambda_j + 2\mu_j} B
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&\widetilde{\lambda} = \eta\lambda_1 + (1-\eta)\lambda_2 && [\lambda] = \eta(\lambda_1 - \lambda_2) \\
&\widehat{\lambda} = \eta\lambda_1 + \frac{\eta^2}{1-\eta}\lambda_2 && \widetilde{\mu} = \eta\mu_1 + (1-\eta)\mu_2 \\
&[\mu] = \eta(\mu_1 - \mu_2) && \widehat{\mu} = \eta\mu_1 + \frac{\eta^2}{1-\eta}\mu_2
\end{aligned}
\tag{2.8}
$$

It should be emphasized that the continuity conditions on the interfaces are satisfied because the relations for stress components $\sigma_{xx}^{(j)}$, $\sigma_{xy}^{(j)}$, $j = 1, 2$ (see Eqs. (2.6)) are independent of the kind of composite constituent. The stresses $\sigma_{yy}^{(j)}$ have jumps on the interfaces, also on the boundary of the half-plane. It leads to a rather complicated boundary value problem described by equations (2.5) with the following boundary conditions

$$
\begin{aligned}
&\sigma_{yy}^{(j)}(x,0) = -P\delta(x-a) && P = \sigma_0 \sin\theta \\
&\sigma_{xy}^{(j)}(x,0) = -Q\delta(x-a) && Q = \sigma_0 \cos\theta
\end{aligned}
\tag{2.9}
$$

and regularity conditions at infinity

$$
\sigma_{xx}^{(j)}, \sigma_{yy}^{(j)}, \sigma_{xy}^{(j)} \to 0 \qquad \text{for} \quad x^2 + y^2 \to \infty
\tag{2.10}
$$

where $\delta(x)$ is the Dirac function and P, Q are known positive constants, see Fig. 1.

Boundary condition $(2.9)_1$ can be approximated by replacing the stress components $\sigma_{yy}^{(j)}$, $j = 1, 2$ by averaged ones as follows

$$
B\frac{\partial U}{\partial x} + A_2\frac{\partial V}{\partial y} = -P\delta(x-a)
\tag{2.11}
$$

It can be shown that

$$
\begin{aligned}
\widetilde{D} &= D_1\eta + D_2(1-\eta) = \frac{\lambda_1\eta}{\lambda_1 + 2\mu_1}A_1 + \frac{\lambda_2(1-\eta)}{\lambda_2 + 2\mu_2}A_2 = B \\
\widetilde{E} &= E_1\eta + E_2(1-\eta) = \eta\Big[\frac{4\mu_1(\lambda_1+\mu_1)}{\lambda_1 + 2\mu_1} + \frac{\lambda_1}{\lambda_1 + 2\mu_1}\Big(\widetilde{\lambda} - \frac{[\lambda]([\lambda] + 2[\mu])}{\widehat{\lambda} + 2\widehat{\mu}}\Big)\Big] + \\
&\quad + (1-\eta)\Big[\frac{4\mu_2(\lambda_2+\mu_2)}{\lambda_2 + 2\mu_2} + \frac{\lambda_2}{\lambda_2 + 2\mu_2}\Big(\widetilde{\lambda} - \frac{[\lambda]([\lambda] + 2[\mu])}{\widehat{\lambda} + 2\widehat{\mu}}\Big)\Big] = A_2
\end{aligned}
$$

Thus, the considered problem is defined by equations (2.5), boundary conditions $(2.9)_2$, (2.11) and regularity conditions (2.10).

3. Solution to the problem

The system of partial differential equations (2.5) can be separated by introducing the following potentials Ψ_1 and Ψ_2 (see Kulchytsky-Zhyhailo *et al.*, 2006)

$$U = \kappa_1 \frac{\partial \Psi_1}{\partial x} + \kappa_2 \frac{\partial \Psi_2}{\partial x} \qquad\qquad V = \frac{\partial \Psi_1}{\partial y} + \frac{\partial \Psi_2}{\partial y} \tag{3.1}$$

where

$$\kappa_i = \frac{A_2\gamma_i^2 - C}{B + C} \qquad\qquad i = 1, 2 \tag{3.2}$$

and γ_i are the roots of the characteristic equation

$$A_2 C\gamma_i^4 + (B^2 + 2BC - A_1A_2)\gamma_i^2 + A_1C = 0 \tag{3.3}$$

Thus, we obtain the following equations for the unknown potentials

$$\gamma_i^2 \frac{\partial^2 \Psi_i}{\partial x^2} + \frac{\partial^2 \Psi_i}{\partial y^2} = 0 \qquad\qquad i = 1, 2 \tag{3.4}$$

Algebraic equation (3.3) in the case of $\mu_1 \neq \mu_2$ has four real roots $\pm\gamma_1$, $\pm\gamma_2$, where

$$\gamma_1 = \sqrt{\frac{A_1A_2 - 2BC - B^2 - \sqrt{\Delta}}{2A_2C}} \qquad\qquad \gamma_2 = \sqrt{\frac{A_1A_2 - 2BC - B^2 + \sqrt{\Delta}}{2A_2C}}$$
$$\Delta = (B^2 + 2BC - A_1A_2)^2 - 4A_1A_2C^2 > 0 \tag{3.5}$$

Let $\widetilde{f}$ denote the Fourier transform of an integrable function f with respect to the variable x

$$\widetilde{f}(s, y) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x, y) \exp(-\mathrm{i}xs)\, dx \tag{3.6}$$

By using equations (3.4), (3.1) and regularity conditions (2.10), we obtain

$$\begin{aligned} \widetilde{U} &= \mathrm{i}s[\kappa_1 a_1(s) \exp(-|s|\gamma_1 y) + \kappa_2 a_2(s) \exp(-|s|\gamma_2 y)] \\ \widetilde{V} &= -|s|[\gamma_1 a_1(s) \exp(-|s|\gamma_1 y) + \gamma_2 a_2(s) \exp(-|s|\gamma_2 y)] \end{aligned} \tag{3.7}$$

where $a_1(s)$, $a_2(s)$ should be determined from boundary conditions $(2.9)_2$, (2.11) together with (2.6). It leads to the following system of algebraic equations

$$\begin{aligned} (A_2\gamma_1^2 - \kappa_1 B)a_1(s) + (A_2\gamma_2^2 - \kappa_2 B)a_2(s) &= \frac{-P\exp(-\mathrm{i}as)}{s^2\sqrt{2\pi}} \\ -\gamma_1(1 + \kappa_1)a_1(s) - \gamma_2(1 + \kappa_2)a_2(s) &= \frac{\mathrm{i}Q\exp(-\mathrm{i}as)}{s|s|\sqrt{2\pi}} \end{aligned} \tag{3.8}$$

The solution to equations (3.8) takes the form

$$
\begin{aligned}
a_1(s) &= \frac{(\gamma_2|s|P - \mathrm{i}sQ)\exp(-\mathrm{i}as)}{|s|s^2(\gamma_1-\gamma_2)\sqrt{2\pi}}\,\frac{B+C}{(A_2\gamma_1^2+B)C} \\
a_2(s) &= \frac{(\mathrm{i}sQ - \gamma_1|s|P)\exp(-\mathrm{i}as)}{|s|s^2(\gamma_1-\gamma_2)\sqrt{2\pi}}\,\frac{B+C}{(A_2\gamma_2^2+B)C}
\end{aligned}
\tag{3.9}
$$

Knowing the functions $a_1(s)$, $a_2(s)$ and using equations (3.1), (2.6), after some calculations of the inverse Fourier transforms for the obtained stress components, we find a closed form of the solution

$$
\begin{aligned}
\sigma_{xx}^{(j)}(x,y) &= (\gamma_1^2 B - \kappa_1 A_1)G_1\frac{Q(x-a)+P\gamma_1\gamma_2 y}{\pi[(a-x)^2+y^2\gamma_1^2]} + \\
&\quad -(\gamma_2^2 B - \kappa_2 A_1)G_2\frac{Q(x-a)+P\gamma_1\gamma_2 y}{\pi[(a-x)^2+y^2\gamma_2^2]} \\
\sigma_{yy}^{(j)}(x,y) &= (\gamma_1^2 E_j - \kappa_1 D_j)G_1\frac{Q(x-a)+P\gamma_1\gamma_2 y}{\pi[(a-x)^2+y^2\gamma_1^2)} + \\
&\quad -(\gamma_2^2 E_j - \kappa_2 D_j)G_2\frac{Q(x-a)+P\gamma_1\gamma_2 y}{\pi[(a-x)^2+y^2\gamma_2^2]} \\
\sigma_{xy}^{(j)}(x,y) &= -\gamma_1(1+\kappa_1)CG_1\frac{Q\gamma_1 y + P(a-x)\gamma_2}{\pi[(a-x)^2+y^2\gamma_1^2]} + \\
&\quad +\gamma_2(1+\kappa_2)CG_2\frac{Q\gamma_2 y + P(a-x)\gamma_1}{\pi[(a-x)^2+y^2\gamma_2^2]}
\end{aligned}
\tag{3.10}
$$

and

$$
G_1 = \frac{1}{\gamma_1-\gamma_2}\,\frac{B+C}{(A_2\gamma_1^2+B)C} \qquad\qquad G_2 = \frac{1}{\gamma_1-\gamma_2}\,\frac{B+C}{(A_2\gamma_2^2+B)C} \tag{3.11}
$$

4. Discussion of the results

Let us introduce the following dimensionless variables

$$
x' = \frac{x}{l} \qquad\qquad y' = \frac{y}{l} \tag{4.1}
$$

The distributions of dimensionless stresses $\sigma_{xx}^{(j)}(x',y')/\sigma_0$ due to the concentrated force acting at the point $(0,0)$ for two special cases

$$
\begin{aligned}
&(1^\circ) \qquad P = \sigma_0 \qquad\qquad Q = 0 \\
&(2^\circ) \qquad P = 0 \qquad\qquad Q = \sigma_0
\end{aligned}
\tag{4.2}
$$

are presented in Figures 2a (case 1°) and 2b (case 2°) for the ratio of Young's moduli $E_1/E_2 = 4$, $\eta = 0.5$ the Poissons ratios $\nu_1 = \nu_2 = 0.3$ and $a = 0$.

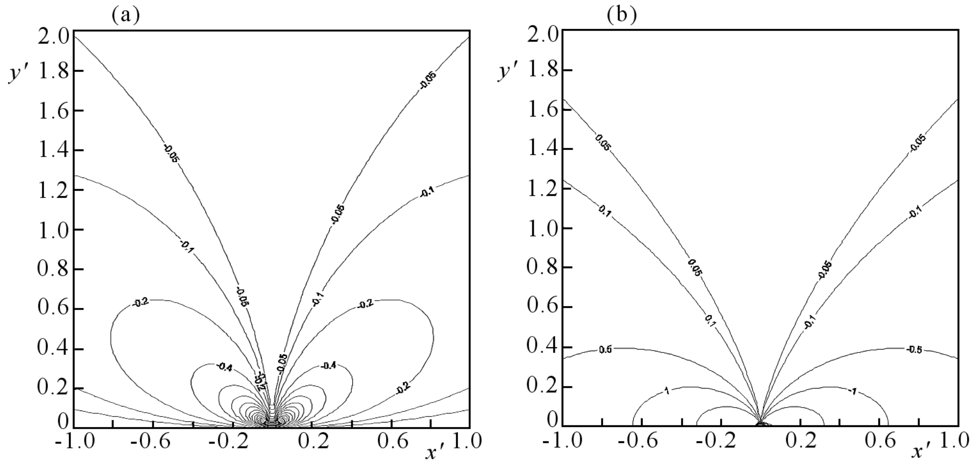

Fig. 2. Lines of constant dimensionless stresses $\sigma_{xx}^{(j)}(x',y')/\sigma_0$ for two cases: (a) $P = \sigma_0$, $Q = 0$, (b) $P = 0$, $Q = \sigma_0$ ($E_1/E_2 = 4$, $\eta = 0.5$, $\nu_1 = \nu_2 = 0.3$, $a = 0$)

Figures 3a and 3b show lines of constant dimensionless stresses of $\sigma_{xy}^{(j)}(x',y')/\sigma_0$ for case (1°) and case (2°), respectively.

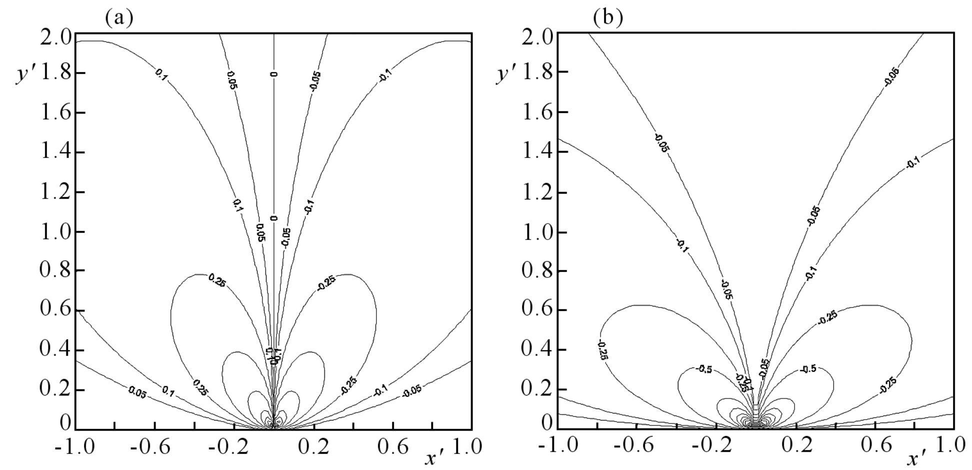

Fig. 3. Lines of constant dimensionless stresses $\sigma_{xy}^{(j)}(x',y')/\sigma_0$ for two cases: (a) $P = \sigma_0$, $Q = 0$, (b) $P = 0$, $Q = \sigma_0$ ($E_1/E_2 = 4$, $\eta = 0.5$, $\nu_1 = \nu_2 = 0.3$, $a = 0$)

The component of stresses $\sigma_{yy}^{(j)}(x',y')/\sigma_0$ are presented in Fig. 4 and Fig. 5 The distributions of $\sigma_{yy}^{(j)}(x',y')/\sigma_0$ for the case (1°), (see Eq. (4.2)) are shown in Fig. 4a in the plane $y' = 0.25$, and in Fig. 4b in the plane $y' = 0.5$. It is seen that the values of jumps on the interfaces decrease together with growing distance from the boundary. Figures 5a,b present distributions of dimensionless stresses $\sigma_{yy}^{(j)}(x',y')/\sigma_0$ for case $(2°)$ for two depths $y' = 0.25$; 0.5, respectively.

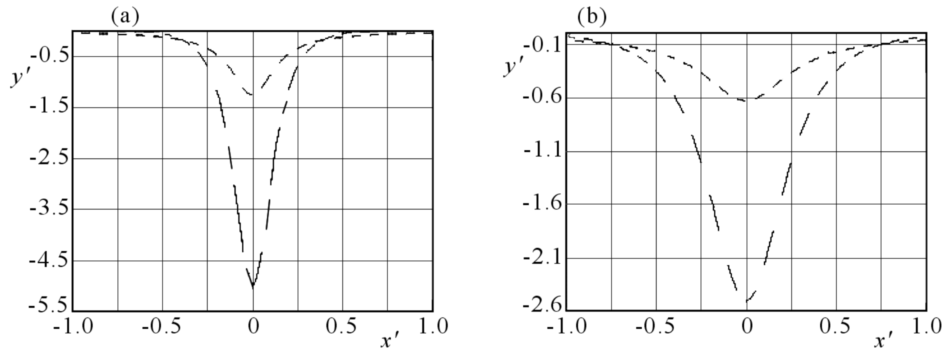

Fig. 4. Lines of constant dimensionless stresses $\sigma_{yy}^{(j)}(x',y')/\sigma_0$ for $P = \sigma_0$, $Q = 0$; (a) $y' = 0.25$, (b) $y' = 0.5$ ($E_1/E_2 = 4$, $\eta = 0.25$, $\nu_1 = \nu_2 = 0.3$, $a = 0$)

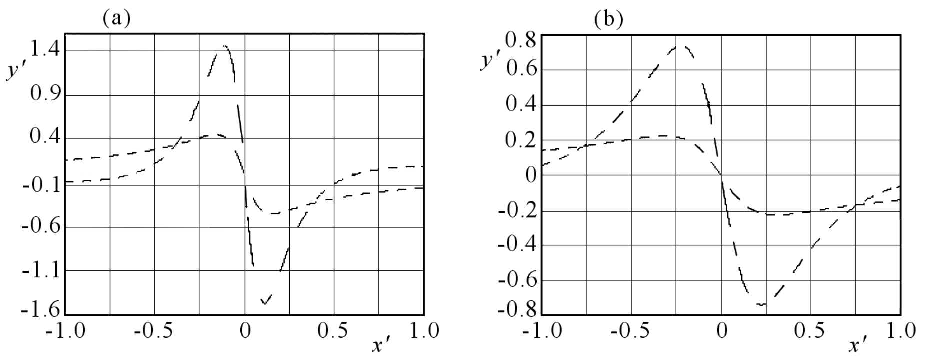

Fig. 5. Lines of constant dimensionless stresses $\sigma_{yy}^{(j)}(x',y')/\sigma_0$ for) $P = 0$, $Q = \sigma_0$; (a) $y' = 0.25$, (b) $y' = 0.5$ ($E_1/E_2 = 4$, $\eta = 0.25$, $\nu_1 = \nu_2 = 0.3$, $a = 0$)

5. Final remarks

The stress distribution given by equations (3.10) stands for the fundamental solution for a periodically laminated half-space with the layering normal to the boundary, and can be used to solve some boundary value problems of a nonhomogeneous body (for example: a crack normal to the layering in the composite space, contact of the half-space with punches). The analysis presented above is strictly connected with the case of shear modulus $\mu_1 \neq \mu_2$. If $\mu_1 = \mu_2$, calculations of stresses should be carried out starting from characteristic equation (3.3). The same investigations were presented for an analogical problem of the fundamental solution for a laminated half-space with the boundary parallel to the layering, see Kaczyńsk and Matysiak (1987).

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Osobliwość naprężeń w periodycznie warstwowej półprzestrzeni z brzegiem prostopadłym do lamin

Streszczenie

W pracy rozpatrzono płaskie zagadnienie dotyczące rozkładu naprężeń w niejednorodnej sprężystej półprzestrzeni wywołanych obciążeniami skupionymi działającymi na jej brzegu. Ośrodek jest złożony z periodycznie powtarzającymi się dwuskładnikowymi laminami, a jego brzeg jest prostopadły do uwarstwienia. Otrzymano rozwiązanie w ramach modelu homogenizowanego z parametrami mikrolokalnymi (Woźniak, 1987; Matysiak i Woźniak, 1987). Zostały podane analityczne wyrażenia dla naprężeń i następnie przedstawione w postaci wykresów.

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