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THE 3D THERMOELASTIC PROBLEM OF UNIFORM HEAT FLOW DISTURBED BY AN INTERFACE CRACK IN A PERIODIC TWO-LAYER COMPOSITE[1]

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This paper is concerned with the problem of an interface insulated plane crack obstructing a uniform heat flux in a two-layer microperiodic space. An approximate analysis is performed within the framework of linear stationary thermoelasticity with microlocal parameters. A general method of solving the resulting boundary-value problem is presented. It is based on the use of potential functions and an analogy between the thermal crack problem and the skew-symmetric mechanical loading problem. The thermal stress singularities are discussed.

Key words: periodic two-layered space, homogenized model, interface crack, heat flow, thermal stresses

1. Introduction

Due to the rapidly increasing use of composites for engineering structures, considerable attention has been given to the analysis of interface cracks subjected to mechanical and thermal loading (see the papers included in the volume edited by Rossmanith (1997)). The well-known conventional solutions, in this case, exhibit peculiar oscillatory singularities near crack borders, which are physically unacceptable.

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The present work aims at the three-dimensional problem of uniform heat flow in a bimaterial periodically layered space disturbed by an interface crack. It is a sequel of our earlier research in the two-dimensional case (cf Kaczyński and Matysiak, 1989, 1998). The considerations are based on the use of the homogenized model of a microperiodic two-layer composite, proposed by Woźniak (1987), Matysiak and Woźniak (1988). This approximate treatment – employed to interface cracks – fails to predict the oscillatory singular behavior, and allows us to apply the classical concepts of fracture mechanics in terms of stress intensity factors.

The investigations are being carried on by the general potential function method devised by Kaczyński (1993, 1994). Effective results are obtained owing to a close resemblance between interface crack problems treated within the used homogenized model and the corresponding problems in a transversely isotropic solid. As to the problem under study, a method of constructing its solution by reducing it to an ordinary problem with shear stresses across the surfaces of the crack is presented.

The determination of thermal stresses induced by cracks subject to uniform heat flow is very important for the study of material failure. There were few reports on the three-dimensional analysis of thermoelastic crack problems. The research in this field can be found in the books by Kassir and Sih (1975), Kit and Khay (1989). Related works, pertinent to the present study but concerned with penny-shaped or elliptical cracks in a homogeneous isotropic or transversely isotropic space, were published by Florence and Goodier (1963), Kassir (1971), Tsai (1983).

2. Governing relations

Consider a microperiodic-laminated space as shown in Fig. 1. A thin repeated fundamental layer of the thickness $\delta = \delta_1 + \delta_2$ is composed of two homogeneous sublayers (denoted by $l = 1$ and $l = 2$), characterized by the Lamé constants λ_l, μ_l, thermal conductivities k_l and coefficients of volume expansions $\beta_l/(\lambda_l + 2\mu_l/3)$. Here and in the sequel, the index $l = 1, 2$ refers to the corresponding sublayers. Referring to the Cartesian coordinate system (x_1, x_2, x_3) with the x_3-axis normal to the layering, denote at the point $\boldsymbol{x} = (x_1, x_2, x_3)$ the temperature (strictly, a deviation of the temperature from the reference state) by θ, the displacement vector by $\boldsymbol{u} = [u_1, u_2, u_3]$ and the components of stresses by σ_{11}, σ_{12}, σ_{22}, σ_{13}, σ_{23}, σ_{33}.

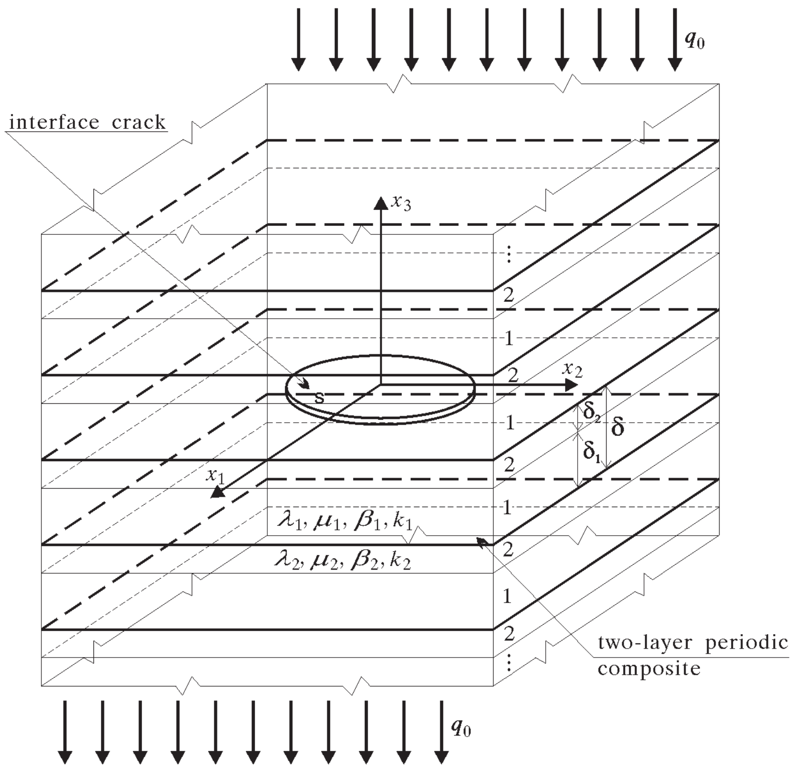

Fig. 1. A uniform heat flow around an insulated interfacial crack in a two-layer periodic space

We suppose that this composite is weakened by an interface crack occupying a bounded plane area S with a smooth profile on the interface $x_3 = 0$, and there is a constant heat flow $\boldsymbol{q}(\infty) = [0, 0, -q_0]$ in the direction of the negative x_3-axis (Fig. 1). The perfect mechanical bonding and ideal thermal contact between the layers (excluding the crack region S) are assumed. The crack surfaces are required to be free from tractions.

Because of the complexity of the composite geometry and boundary conditions, a closed solution to the above crack-thermal stress problem cannot be obtained. Therefore, a specific homogenization procedure called microlocal modelling (Woźniak, 1987; Matysiak and Woźniak, 1988; Kaczyński, 1993) leading to a homogenized model of this layered composite will be employed in order to find an approximate solution. Next, we recall only a brief outline of governing relations of this model; see the papers cited above for details. The following notation will be used: latin subscripts always assume values 1,2,3, and the Greek ones 1,2. The Einstein summation convention holds and subscripts preceded by a comma indicate partial differentiation with respect to the corresponding coordinates.

The following approximations for the temperature θ, displacements u_i, stresses $\sigma_{ij}^{(l)}$ and fluxes $q_i^{(l)}$ constitute the foundations of this approach

$$\begin{aligned}
&\theta \cong \vartheta \qquad\qquad \theta_{,\alpha} \cong \vartheta_{,\alpha} \qquad\qquad \theta_{,3}^{(l)} \cong \vartheta_{,3} + h^{(l)}\Gamma\\
&u_i \cong w_i \qquad\qquad u_{i,\alpha} \cong w_{i,\alpha} \qquad\qquad u_{i,3}^{(l)} \cong w_{i,3} + h^{(l)}d_i\\
&\sigma_{\alpha\beta}^{(l)} \cong \mu_l(w_{\alpha,\beta} + w_{\beta,\alpha}) + \delta_{\alpha\beta}[\lambda_l(w_{i,i} + h^{(l)}d_3) - \beta_l\vartheta]\\
&\sigma_{\alpha 3}^{(l)} \cong \mu_l(w_{\alpha,3} + w_{3,\alpha} + h^{(l)}d_3)\\
&\sigma_{33}^{(l)} \cong (\lambda_l + 2\mu_l)(w_{3,3} + h^{(l)}d_3) + \lambda_l w_{\alpha,\alpha} - \beta_l\vartheta\\
&q_\alpha^{(l)} \cong -k_l\vartheta_{,\alpha} \qquad\qquad q_3^{(l)} \cong -k_l(\vartheta_{,3} + h^{(l)}\Gamma_{,3})
\end{aligned} \tag{2.1}$$

Here, $\delta_{\alpha\beta}$ is the Kronecker delta and $h^{(l)}$ is the derivative of the assumed δ-periodic sectional shape function that becomes: 1 if $l = 1$ ($\boldsymbol{x} \in$ 1st sublayer) and $-\eta/(1-\eta)$ with $\eta = \delta_1/\delta$ if $l = 2$ ($\boldsymbol{x} \in$ 2nd sublayer). Moreover, ϑ, w_i and d_i, Γ are unknown functions interpreted as macro-temperature, macro-displacements and microlocal (thermal and elastic) parameters, respectively.

Applying the microlocal procedure to the macro-modelling of this bimaterial periodically layered composite under stationary conditions, one arrives at governing equations and constitutive relations of the homogenized model. They are given (after eliminating all microlocal parameters and in the absence of body forces and heat sources) in terms of the unknown macro-temperature ϑ and macro-displacements w_i as follows (see Kaczyński, 1994)

$$\vartheta_{,\alpha\alpha} + k_0^{-2}\vartheta_{,33} = 0 \tag{2.2}$$

$$\begin{aligned}
&\frac{1}{2}(c_{11} + c_{12})w_{\beta,\beta\alpha} + \frac{1}{2}(c_{11} - c_{12})w_{\alpha,\beta\beta} + c_{44}w_{\alpha,33} + (c_{13} + c_{44})w_{3,3\alpha} = K_1\vartheta_{,\alpha}\\
&(c_{13} + c_{44})w_{\alpha,\alpha 3} + c_{44}w_{3,\alpha\alpha} + c_{33}w_{3,33} = K_3\vartheta_{,3}
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
\sigma_{\alpha 3}^{(l)} &= c_{44}(w_{\alpha,3} + w_{3,\alpha})\\
\sigma_{33}^{(l)} &= c_{13}w_{\alpha,\alpha} + c_{33}w_{3,3} - K_3\vartheta\\
\sigma_{12}^{(l)} &= \mu_l(w_{1,2} + w_{2,1})\\
\sigma_{11}^{(l)} &= d_{11}^{(l)}w_{1,1} + d_{12}^{(l)}w_{2,2} + d_{13}^{(l)}w_{3,3} - K_2^{(l)}\vartheta\\
\sigma_{22}^{(l)} &= d_{12}^{(l)}w_{1,1} + d_{11}^{(l)}w_{2,2} + d_{13}^{(l)}w_{3,3} - K_2^{(l)}\vartheta
\end{aligned} \tag{2.4}$$

$$q_\alpha^{(l)} = -k_l\vartheta_{,\alpha} \qquad\qquad q_3^{(l)} = -K\vartheta_{,3} \tag{2.5}$$

The positive constants appearing in the above equations, describing material and geometrical characteristics of the composite constituents, are given in Appendix. Observe that the condition of the perfect bonding between the layers is satisfied, and setting $\mu_1 = \mu_2 \equiv \mu$, $\lambda_1 = \lambda_2 \equiv \lambda$ and $\beta_1 = \beta_2 \equiv \beta$, $k_1 = k_2 \equiv k$ we get $c_{11} = c_{33} = \lambda+2\mu$, $c_{12} = c_{13} = \lambda$, $c_{44} = \mu$, $K_1 = K_3 = \beta$, $K = k$, $k_0 = 1$, passing directly to the well-known equations of stationary thermoelasticity of a homogeneous isotropic body.

3. Mathematical formulation and method of solving to the boundary-value problem

Within the scope of the above-presented homogenized model, we deal with the resulting boundary-value problem: find fields ϑ and w_i, σ_{ij} suitable smooth on $R^3 - S$ such that Eqs (2.2)-(2.5) hold, subject to the following boundary conditions

$$\left.\begin{array}{l} q_3 = -K\vartheta_{,3} = 0 \\ \sigma_{i3} = 0 \end{array}\right\} \qquad \forall(x_1, x_2, x_3 = 0^{\pm}) \in S$$
$$\left.\begin{array}{l} q_3 = -K\vartheta_{,3} = -q_0 \\ \sigma_{i3} = 0 \end{array}\right\} \qquad \text{if} \ \sqrt{x_1^2 + x_2^2 + x_3^2} \to \infty \tag{3.1}$$

Making use of the superposition principle, we construct the solution as a sum of two parts – the state of simple flow of uniform heat in the uncracked medium and the perturbation temperature and stress field due to the crack that tends to zero at large distances from the origin.

The first 0-problem involves the solution to basic equations (2.2) with conditions $(3.1)_2$. The results are found to be

$$\begin{aligned} \vartheta^0(x_1, x_2, x_3) &= \frac{q_0}{K}x_3 \\ \boldsymbol{w}^0(x_1, x_2, x_3) &= \frac{K_3 q_0}{K(2c_{13} + c_{33})}\Big[x_1x_3, x_2x_3, \frac{1}{2}(x_3^2 - x_1^2 - x_2^2)\Big] \\ \sigma_{i3}^0(x_1, x_2, x_3) &= 0 \end{aligned} \tag{3.2}$$

Attention will be drawn then on the corrective solution to the perturbed problem (designated by the tilde). The disturbance due to the crack S results in a macro-temperature $\widetilde{\vartheta}$ and induced thermal stresses $\widetilde{\sigma}_{ij}$. The steady-state temperature field is first determined. In view of the skew-symmetry, the problem is reduced to that of finding $\widetilde{\vartheta}$ satisfying so-called quasi-Laplace's equation (2.2) in the half space $x_3 \geqslant 0$, subject to the boundary conditions

$$\begin{aligned} &\widetilde{\vartheta}_{,3}\Big|_{S^+} = -\frac{q_0}{K} \qquad\qquad \widetilde{\vartheta}\Big|_{R^2-S^+} = 0 \\ &\widetilde{\vartheta} \to 0 \qquad \text{for} \qquad \sqrt{x_1^2+x_2^2+x_3^2} \to \infty \end{aligned} \tag{3.3}$$

It is deducted from the potential theory that the solution of this problem can be written as

$$\widetilde{\vartheta}(x_1,x_2,x_3) = \frac{\partial}{\partial z_0} \iint\limits_S \frac{\gamma(\xi_1,\xi_2)\; d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+z_0^2}} \tag{3.4}$$

where $z_0 = k_0 x_3$ and γ is an unknown density that satisfies the integro-differential singular equation of Newton's potential type

$$\nabla^2 \iint\limits_S \frac{\gamma(\xi_1,\xi_2)\; d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} = \frac{q_0}{K k_0} \tag{3.5}$$

in which $\nabla^2 \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ stands for the two-dimensional Laplace operator.

We proceed now to an associated thermoelastic problem. Because of the anti-symmetry of the stress system, it reduces to that of the half space $x_3 \geqslant 0$ subjected to the skew-symmetry and stress-free conditions

$$\begin{aligned} &\widetilde{\sigma}_{33}(x_1,x_2,x_3=0^+) = 0 && \forall(x_1,x_2) \in R^2 \\ &\widetilde{w}_\alpha(x_1,x_2,x_3=0^+) = 0 && \forall(x_1,x_2) \in R^2 - S \\ &\widetilde{\sigma}_{3\alpha}(x_1,x_2,x_3=0^+) = 0 && \forall(x_1,x_2) \in S \end{aligned} \tag{3.6}$$

An efficient approach is based on the construction of quasi-harmonic potentials that satisfy governing equations (2.3) and are well suited to mixed boundary conditions (3.6). Utilizing the results derived by Kaczyński (1994), the perturbed problem reduces to the finding of two harmonic functions g and h which satisfy the following boundary conditions resulting from Eqs (3.6)

— for $(x_1,x_2) \in S$

$$\begin{aligned} g_{,33} + \widehat{\nu}(g_{,22} - h_{,12}) &= \widehat{\beta}\Big[\iint\limits_S \frac{\gamma(\xi_1,\xi_2)\; d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}}\Big]_{,1} \\ h_{,33} + \widehat{\nu}(h_{,11} - g_{,12}) &= \widehat{\beta}\Big[\iint\limits_S \frac{\gamma(\xi_1,\xi_2)\; d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}}\Big]_{,2} \end{aligned} \tag{3.7}$$

— for $(x_1,x_2) \in R^2 - S$

$$g_{,3} = h_{,3} = 0 \tag{3.8}$$

with the constants $\widehat{\nu}$ and $\widehat{\beta}$ given in Appendix.

These conditions are seen to be similar to those for shear crack mechanical loadings given by the right-hand sides of equations (3.7), in an isothermal elasticity state (cf Kassir and Sih, 1975). Thus, the thermal crack problem in hand is reduced to its mechanical skew-symmetrical counterpart provided we are able to solve Eq. (3.5) and evaluate the corresponding derivatives of potentials appearing in (3.7). Exact results can be obtained for a crack S in the shape of an ellipse. For illustration, we present them in the case of a circular interface crack with the radius a, i.e. $S_a = \{(x_1,x_2,0) : r \equiv \sqrt{x_1^2+x_2^2} \leqslant a\}$. Then, the solution to Eq. (3.5) is (see Kit and Khay, 1989)

$$\gamma(x_1,x_2) = -\frac{q_0}{\pi^2 K k_0}\sqrt{a^2-x_1^2-x_2^2} \qquad \forall (x_1,x_2) \in S_a \tag{3.9}$$

and the derivatives of the corresponding potentials may be evaluated such that the perturbed problem described by (3.7) and (3.8) takes the form

— for $(x_1,x_2) \in S_a$

$$\begin{aligned} g_{,33} + \widehat{\nu}(g_{,22} - h_{,12}) &= \frac{\widehat{\beta} q_0}{2Kk_0}x_1 \\ h_{,33} + \widehat{\nu}(h_{,22} - g_{,12}) &= \frac{\widehat{\beta} q_0}{2Kk_0}x_2 \end{aligned} \tag{3.10}$$

— for $(x_1,x_2) \in R^2 - S_a$

$$g_{,3} = h_{,3} = 0 \tag{3.11}$$

It can be observed that this corresponds to the ordinary boundary space problem of radial shear in an isotropic case, when the stresses on the crack surfaces S_a are given as

$$\sigma_{3r} = -\frac{\widehat{\beta} q_0}{2Kk_0\kappa}r \qquad x_3 = 0 \qquad r \leqslant a \tag{3.12}$$

where κ is a constant defined in Appendix.

Hence, the potential functions g and h leading to the stress distribution in the laminated medium can be found from the solution to this problem which is known (see, for example, Kassir and Sih, 1975). The singular behavior of the interface crack-border thermal stresses is similar to that in the homogeneous case of transversely isotropic bodies. The shearing stress σ_{3r} near the crack border possesses the inverse square-root type the singularity, and is characterized by the stress intensity factor given by

$$k_{II} \equiv \lim_{r\to a^+} \sqrt{2(r-a)}\,\sigma_{3r}(r,0) = \frac{2a\sqrt{a}q_0\widehat{\beta}}{3\pi K k_0 \kappa} \tag{3.13}$$

A physically interesting observation is the influence of layering of the considered body on this factor, expressed by the term $\widehat{\beta}/(Kk_0\kappa)$.

Appendix

Denoting by

$$B_l = \lambda_l + 2\mu_l \qquad\qquad l = 1,2$$

$$\overline{B} = (1-\eta)B_1 + \eta B_2 \qquad\qquad \overline{K} = (1-\eta)k_1 + \eta k_2$$

the positive coefficients in governing equations (2.2)-(2.5) are given by the following formulae

$$k_0 = \sqrt{\frac{\eta k_1\overline{K} + (1-\eta)k_2\overline{K}}{k_1k_2}} \qquad\qquad c_{33} = \frac{B_1B_2}{\overline{B}}$$

$$c_{11} = c_{33} + \frac{4\eta(1-\eta)(\mu_1-\mu_2)(\lambda_1-\lambda_2+\mu_1-\mu_2)}{\overline{B}}$$

$$c_{12} = \frac{\lambda_1\lambda_2 + 2[\eta\mu_2 + (1-\eta)\mu_1][\eta\lambda_1 + (1-\eta)\lambda_2]}{\overline{B}}$$

$$c_{13} = \frac{(1-\eta)\lambda_2B_1 + \eta\lambda_1B_2}{\overline{B}} \qquad\qquad c_{44} = \frac{\mu_1\mu_2}{(1-\eta)\mu_1 + \eta\mu_2}$$

$$K_1 = \frac{\eta\beta_1\lambda_2 + (1-\eta)\beta_2\lambda_1 + 2[(1-\eta)\mu_1 + \eta\mu_2][\eta\beta_1 + (1-\eta)\beta_2]}{\overline{B}}$$

$$K_3 = \frac{(1-\eta)\beta_2B_1 + \eta\beta_1B_2}{\overline{B}} \qquad\qquad K = \frac{k_1k_2}{\overline{K}}$$

$$K_2^{(l)} = \frac{2\mu_l\beta_l + \lambda_lK_3}{B_l} \qquad\qquad d_{11}^{(l)} = \frac{4\mu_l(\lambda_l+\mu_l) + \lambda_lc_{13}}{B_l}$$

$$d_{12}^{(l)} = \frac{2\mu_l\lambda_l + \lambda_lc_{13}}{B_l} \qquad\qquad d_{13}^{(l)} = \frac{\lambda_lc_{33}}{B_l}$$

The constants in Eqs (3.12), (3.7) and (3.8) are derived from Kaczyński (1994), and are given as follows

$$\kappa = \frac{t_+ c_{33}}{c_{11}c_{33} - c_{13}^2} \qquad \widehat{\nu} = 1 - \kappa t_3 c_{44}$$

$$\widehat{\beta} = c_1 + \frac{K_3 - c_1 c_{13} - c_2 c_{33} k_0^2}{\sqrt{c_{11}c_{33}} + c_{13}} - c_{44} k_0 \kappa (c_1 - c_2)$$

provided

$$t_+ = \sqrt{\frac{c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}}{c_{33}c_{44}} + 2\sqrt{\frac{c_{11}}{c_{33}}}}$$

$$t_3 = \sqrt{\frac{\eta\mu_1 + (1-\eta)\mu_2}{c_{44}}}$$

$$c_1 = \frac{k_0^2[(c_{13} + c_{44})K_3 - c_{33}K_1] + c_{44}K_1}{c_{33}c_{44}k_0^4 + (c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33})k_0^2 + c_{11}c_{44}}$$

$$c_2 = \frac{k_0[(c_{13} + c_{44})K_3 - c_{11}K_3 + k_0^2 c_{44} K_3]}{c_{33}c_{44}k_0^4 + (c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33})k_0^2 + c_{11}c_{44}}$$

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Trójwymiarowe zagadnienie stałego przepływu ciepła zakłóconego przez międzywarstwową szczelinę w periodycznym dwuwarstwowym kompozycie

Streszczenie

Problem stałego przepływu ciepła zakłóconego istnieniem izolowanej termicznie szczeliny międzywarstwowej w periodycznej dwuwarstwowej przestrzeni jest przedmiotem pracy. Przybliżoną analizę przeprowadzono w ramach liniowej stacjonarnej termosprężystości z parametrami mikrolokalnymi. Podano i zilustrowano metodę rozwiązania wynikającego zagadnienia brzegowego, polegającą na zastosowaniu potencjałów harmonicznych i ustaleniu analogii z odpowiadającym mechanicznym, niesymetrycznym problemem szczeliny. Zbadano osobliwości naprężeń cieplnych wokół szczeliny.

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