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# A SIMPLICIAL MODEL FOR DYNAMIC PROBLEMS IN PERIODIC MEDIA

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The aim of contribution is to specify a class of periodic mass-point systems with ternary interactions which can be interpreted as discrete models of plane dynamic problems in linear-elastic composite materials. The obtained results yield a new mathematical tool for the analysis of wave propagation problems in heterogeneous periodic material structures, which can be carried out on different levels of accuracy. An application of the proposed model to the analysis of a plane wave propagating in the homogeneous medium with prizmatic rigid inclusions is shown.

*Key words:* composites, modelling, dynamics

## 1. Introduction

A list of papers on dynamic problems in material continua with a periodic structure is very extensive and will be not discussed here. A survey of earlier contributions, mainly related to the wave propagation problems can be found in Lee (1972). In most cases the exact analysis of these problems is not possible even using computer methods. That is why different approximate mathematical models of dynamic phenomena in heterogeneous periodic media have been proposed, Achenbach et al. (1968), Herrmann and Achenbach (1968), Sun et al. (1968). In this contribution we are to show that two-dimensional dynamic problems in periodic composite materials can be modelled by certain discrete plane periodic mass-point systems with a complex structure and ternary interactions. To this end we begin with a formulation of governing equations for the above systems by applying a certain generalization of the approach proposed in Rychlewska et al. (1999). It will be shown that this approach makes it possible to formulate discrete models of dynamic problems in composite materials on different levels of accuracy. For the sake of simplicity considerations will be restricted to the linear-elastic material structures and plane dynamic problems for an unbounded medium. General considerations will be illustrated by the analysis of a plane wave propagation in the homogeneous isotropic periodic linear-elastic medium with two kinds of rigid inclusions.

**Notations.** The superscripts $a, b, c$ run over $1, ..., n$ and the superscript $k$ takes the values $1, ..., m$. Indices $A, B, C$ run over $0, 1, ..., N$ except in denotations $\Delta_A, \overline{\Delta}_A$, where $A = 1, ..., N$ unless otherwise stated. Summation convention holds for all the indices repeated twice. Points on $E^2$ are denoted by $\boldsymbol{p}$, $\boldsymbol{x}$ and points belonging to a subset $\Lambda$ of $E^2$ by $\boldsymbol{z}$. The symbol $t$ stands for a time coordinate.

## 2. Preliminaries

Let $[\boldsymbol{d}^1, \boldsymbol{d}^2]$ be a vector basis on $E^2$ and $\Lambda$ stand for the Bravais lattice

$$\Lambda := \left\{ \boldsymbol{z} \in E^2 : \quad \boldsymbol{z} = \nu_1 \boldsymbol{d}^1 + \nu_2 \boldsymbol{d}^2, \quad \nu_\alpha = 0, \pm 1, \pm 2, ..., \quad \alpha = 1, 2 \right\}$$

For an arbitrary subset $\Xi$ of $E^2$ and every $\boldsymbol{z} \in \Lambda$ define $\Xi(\boldsymbol{z}) \equiv \Xi + \boldsymbol{z}$. Similarly define $\boldsymbol{p}(\boldsymbol{z}) \equiv \boldsymbol{p} + \boldsymbol{z}$ for any $\boldsymbol{p} \in E^2$ and $\boldsymbol{z} \in \Lambda$. Let $\Delta$ be a regular region on $E^2$ such that $E^2 = \bigcup \overline{\Delta}(\boldsymbol{z})$, $\boldsymbol{z} \in \Lambda$ and $\Delta(\boldsymbol{z}_1) \cap \Delta(\boldsymbol{z}_2) = \emptyset$ for every $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \Lambda$ and $\boldsymbol{z}_1 \neq \boldsymbol{z}_2$. We shall also assume that there exist a simplicial subdivision of $\Delta$ into $m$ simplexes $T^k$, $k = 1, ..., m$, which implies the simplicial subdivision of $E^2$ into simplexes $T^k(\boldsymbol{z})$, $\boldsymbol{z} \in \Lambda$. Hence $\bigcup \overline{T}^k = \overline{\Delta}$ and

$$T := \left\{ T^k(\boldsymbol{z}) : \quad \boldsymbol{z} \in \Lambda, \quad k = 1, ..., m \right\} \tag{2.1}$$

constitutes a set of all simplexes for the subdivision of $E^2$. It can be seen that for the aforementioned simplicial subdivision of $\Lambda$ there exist a set of vertices $\boldsymbol{p}^a \in \overline{\Delta}$, $a = 1, ..., n$, such that

$$S := \left\{ \boldsymbol{p}^a(\boldsymbol{z}) : \quad \boldsymbol{z} \in \Lambda, \quad a = 1, ..., n \right\} \tag{2.2}$$

is a set of all vertices for the related subdivision of $E^2$. In the sequel we shall assume that $n$ is the smallest number of vertices $\boldsymbol{p}^a \in \overline{\Delta}$ for which $S$ is a set of all vertices and that the decomposition of $\boldsymbol{p}^a(\boldsymbol{z})$ in the form $\boldsymbol{p}^a(\boldsymbol{z}) = \boldsymbol{p}^a + \boldsymbol{z}$, $\boldsymbol{z} \in \Lambda$, $a = 1, ..., n$, is unique. In the subsequent considerations both simplicial subdivision of $\Delta$ and a set of $n$ vertices $\boldsymbol{p}^a$, $a = 1, ..., n$, are assumed to be known.

Let $\boldsymbol{d}^A \in \overline{\Delta}$, $A = 0, 1, ..., N$, be a system of vectors where $\boldsymbol{d}^0 = \boldsymbol{0}$ and $\boldsymbol{p}^a$, $a = 1, ..., n$ a set of vertices such that all simplex vertices belonging to $\overline{\Delta}$ can be uniquely represented in the form $\boldsymbol{p}^a_A = \boldsymbol{p}^a + \boldsymbol{d}^A$. Hence every $T^k$ can be represented as $T^k = \boldsymbol{p}^a_A \boldsymbol{p}^b_B \boldsymbol{p}^c_C$. For an arbitrary function $f(\cdot)$ defined on $S$ with values in a certain linear space we shall introduce the finite differences

$$\Delta_A f(\boldsymbol{z}) \equiv f(\boldsymbol{z} + \boldsymbol{d}^A) - f(\boldsymbol{z}) \qquad\qquad \overline{\Delta}_A f(\boldsymbol{z}) \equiv f(\boldsymbol{z}) - f(\boldsymbol{z} - \boldsymbol{d}^A)$$

which for $A = 0$ reduce to identities. Here and in the sequel all finite difference operators $\Delta_A, \overline{\Delta}_A$ will be defined only for $A = 1, ..., N$. Following the notation introduced above we also denote $T^k(\boldsymbol{z}) \equiv T^k + \boldsymbol{z}$ and $\boldsymbol{p}^a_A(\boldsymbol{z}) = \boldsymbol{p}^a_A + \boldsymbol{z}$ for every $\boldsymbol{z} \in \Lambda$.

The aforementioned concepts and definitions will be used in the next section in order to specify a certain class of periodic mass-point systems with ternary interactions.

## 3. Periodic mass-point systems with ternary interactions

The analysis of any mass-point system requires the use of a certain parametrization of this system. In the problem under consideration it will be assumed that the position of mass-point system in its reference equilibrium state coincides with a set $S$ of points in $E^2$. Hence every mass point will be parametrized by its reference position $\boldsymbol{p}^a(\boldsymbol{z})$, $\boldsymbol{z} \in \Lambda$, $a = 1, ..., n$. Displacements of these points from their reference positions at a time $t$ will be denoted by $\boldsymbol{u}^a(\boldsymbol{z}, t)$. It is assumed that to every point $\boldsymbol{p}^a(\boldsymbol{z}) \in S$ there is assigned mass $m^a$ which due to the periodicity of system is independent of $\boldsymbol{z} \in \Lambda$. An external force acting at $\boldsymbol{p}^a(\boldsymbol{z})$ at time $t$ will be denoted by $\boldsymbol{f}^a(\boldsymbol{z}, t)$. The system of ternary interactions will be parametrized by a set $T$ of all simplexes by assuming that the points $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in S$ can interact if and only if $\boldsymbol{x}_1 = \boldsymbol{p}^a_A(\boldsymbol{z})$, $\boldsymbol{x}_2 = \boldsymbol{p}^b_B(\boldsymbol{z})$, $\boldsymbol{x}_3 = \boldsymbol{p}^c_C(\boldsymbol{z})$ for some $\boldsymbol{z} \in \Lambda$, where $\boldsymbol{p}^a_A(\boldsymbol{z})\boldsymbol{p}^b_B(\boldsymbol{z})\boldsymbol{p}^c_C(\boldsymbol{z}) = T^k(\boldsymbol{z})$ for some $k \in \{1, ..., m\}$. Hence every ternary interaction will be identified with a certain simplex $T^k(\boldsymbol{z})$, $\boldsymbol{z} \in \Lambda$, $k = 1, ..., m$. It is also assumed that to every interaction $T^k(\boldsymbol{z}) \in T$ there is assigned a strain energy function $\Phi^k$ which by means of periodicity of the system is independent of $\boldsymbol{z} \in \Lambda$. Let $\boldsymbol{u}^a_A(\boldsymbol{z}, t)$ stand for a displacement vector of point $\boldsymbol{p}^a_A(\boldsymbol{z})$ at time $t$. Under these denotation the arguments of $\Phi^k$ are $|\boldsymbol{u}^a_A(\boldsymbol{z},t) - \boldsymbol{u}^b_B(\boldsymbol{z},t)|$. Because of

$$\boldsymbol{u}^a_A(\boldsymbol{z},t) = \boldsymbol{u}^a(\boldsymbol{z}+\boldsymbol{d}^A,t) - \boldsymbol{u}^a(\boldsymbol{z},t) + \boldsymbol{u}^a(\boldsymbol{z},t) = \Delta_A\boldsymbol{u}^a(\boldsymbol{z},t) + \boldsymbol{u}^a(\boldsymbol{z},t)$$

we shall assume that

$$\Phi^k = \Phi^k\Big(\Delta_A\boldsymbol{u}^a(\boldsymbol{z},t), \boldsymbol{u}^b(\boldsymbol{z},t) - \boldsymbol{u}^c(\boldsymbol{z},t)\Big) \tag{3.1}$$

bearing in mind that $\Phi^k(\cdot)$ are hemitropic functions of all arguments which are specified by the simplex $T^k = \boldsymbol{p}^a_A\boldsymbol{p}^b_B\boldsymbol{p}^c_C$.

It can be seen that to every $\boldsymbol{z} \in \Lambda$ there is assigned a certain repetitive element of the periodic system under consideration comprising $n$ mass points $\boldsymbol{p}^a(\boldsymbol{z})$, $a = 1,...,n$, and $m$ ternary interactions $T^k(\boldsymbol{z})$, $k = 1,...,m$. Define $M^{ab} = \delta^{ab}m^b$ (no summation over $b$). The kinetic and strain energy functions assigned to an arbitrary repetitive element are respectively given by

$$\begin{aligned} K &= \frac{1}{2}M^{ab}\dot{\boldsymbol{u}}^a(\boldsymbol{z},t)\dot{\boldsymbol{u}}^b(\boldsymbol{z},t) \\ \Phi &= \sum \Phi^k\Big(\Delta_A\boldsymbol{u}^a(\boldsymbol{z},t), \boldsymbol{u}^b(\boldsymbol{z},t) - \boldsymbol{u}^c(\boldsymbol{z},t)\Big) \end{aligned} \tag{3.2}$$

Using the approach detailed by Woźniak (1971) it can be shown that the above formulae lead to the following equations of motion

$$\overline{\Delta}_A\boldsymbol{S}^a_A(\boldsymbol{z},t) + \boldsymbol{h}^a(\boldsymbol{z},t) - M^{ab}\ddot{\boldsymbol{u}}^b(\boldsymbol{z},t) + \boldsymbol{f}^a(\boldsymbol{z},t) = \boldsymbol{0} \tag{3.3}$$

where $\boldsymbol{S}^a_A$, $\boldsymbol{h}^a$ are generalized internal forces defined by the constitutive equations

$$\boldsymbol{S}^a_A(\boldsymbol{z},t) = \frac{\partial\Phi}{\partial\Delta_A\boldsymbol{u}^a(\boldsymbol{z},t)} \qquad\qquad \boldsymbol{h}^a(\boldsymbol{z},t) = -\frac{\partial\Phi}{\partial\boldsymbol{u}^a(\boldsymbol{z},t)} \tag{3.4}$$

Eqs (3.3), (3.4) have to hold for an arbitrary instant $t$ and every $\boldsymbol{z} \in \Lambda$. They constitute the system of finite difference equations for $\boldsymbol{u}^a(\boldsymbol{z},t)$ involving the second-order time derivatives describing the periodic mass-point system under consideration.

## 4. Simplicial models of periodic composites

Now we are to show that Eqs (3.3), (3.4) can be interpreted as a certain finite element approximation of plane dynamic problems in periodic composite materials. To this end let $E^2$ now represent a two-dimensional linear-elastic continuum with a periodic piecewise homogeneous material structure. Moreover, let $\Delta$ be interpreted as a representative element of this structure and let a simplicial subdivision of $\Delta$ be treated as a decomposition of $\Delta$ into $m$ finite triangle elements $T^k$. At the same time it has to be assumed that every element $T^k$ with a sufficient accuracy has to be treated as homogeneous. Let $T^k = \boldsymbol{p}_A^a \boldsymbol{p}_B^b \boldsymbol{p}_C^c$ and denote by $\lambda^\alpha$, $\alpha = 1,2,3$, the barycentric coordinates of an arbitrary point $\boldsymbol{x} \in T^k$; hence $\lambda^\alpha > 0$, $\lambda^1 + \lambda^2 + \lambda^3 = 1$ and $\lambda^\alpha = \boldsymbol{a}^\alpha \boldsymbol{x} + b^\alpha$ where $\boldsymbol{a}^\alpha \in E^2$ are the known vectors and $b^\alpha$ are the known scalars, Zienkiewicz (1971). The displacement $\boldsymbol{u}(\boldsymbol{x}, t)$ of an arbitrary point $\boldsymbol{x} \in T^k$ at an instant $t$ will be taken in the well known form

$$\boldsymbol{u}(\boldsymbol{x}, t) = \lambda^1 \boldsymbol{u}_A^a(t) + \lambda^2 \boldsymbol{u}_B^b(t) + \lambda^3 \boldsymbol{u}_C^c(t)$$

where $\boldsymbol{u}_A^a(t), \boldsymbol{u}_B^b(t), \boldsymbol{u}_C^c(t)$ are displacements of vertices $\boldsymbol{p}_A^a, \boldsymbol{p}_B^b, \boldsymbol{p}_C^c$, respectively, and $\lambda^\alpha$, $\alpha = 1,2,3$, are the shape functions. The displacement gradient $\nabla \boldsymbol{u}$ is equal to

$$\nabla \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{a}^1 \otimes \boldsymbol{u}_A^a(t) + \boldsymbol{a}^2 \otimes \boldsymbol{u}_B^b(t) + \boldsymbol{a}^3 \otimes \boldsymbol{u}_C^c(t) \qquad \boldsymbol{x} \in T^k \tag{4.1}$$

and hence constant for every instant $t$. Substituting into Eq (4.1) the decompositions of the form

$$\boldsymbol{u}_A^a(t) = \boldsymbol{u}^a(t) + \Delta_A \boldsymbol{u}^a(t) \qquad A = 0, 1, ..., N \tag{4.2}$$

where $\boldsymbol{u}^a(t)$ is a displacement of mass point $\boldsymbol{p}^a \in T$, and introducing the linearized strain tensor

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top) \tag{4.3}$$

we obtain the strain energy function assigned to $T^k$ defined by

$$\Phi^k = \frac{1}{2} F^k \boldsymbol{\varepsilon} : \mathsf{C}_k : \boldsymbol{\varepsilon} \tag{4.4}$$

where $F^k$ is the area of $T^k$ and $\mathsf{C}_k$ is the elastic moduli tensor related to the material of finite element $T^k$. Combining Eqs (4.1) $\div$ (4.4) it can be seen that Eq (4.4) makes it possible to derive the function

$$\Phi^k = \Phi^k(\Delta_A \boldsymbol{u}^a, \boldsymbol{u}^b - \boldsymbol{u}^c) \tag{4.5}$$

depending on differences $\Delta_A \boldsymbol{u}^a$ and $\boldsymbol{u}^b - \boldsymbol{u}^c$. At the same time a constant continuous mass distribution in every $T^k(\boldsymbol{z})$, $\boldsymbol{z} \in \Lambda$, will be replaced by three concentrated masses at vertices $\boldsymbol{p}_A^a(\boldsymbol{z}), \boldsymbol{p}_B^b(\boldsymbol{z}), \boldsymbol{p}_C^c(\boldsymbol{z})$, where $T^k = \boldsymbol{p}_A^a \boldsymbol{p}_B^b \boldsymbol{p}_C^c$.

The above approach is similar to that used in the finite element method and makes it possible to define the concentrated masses $m^a$ at the points $\boldsymbol{p}^a$. From

$$\Phi = \sum \Phi^k \tag{4.6}$$

and bearing in mind Eq (4.5) we obtain the form of strain energy function $\Phi$ assigned to the representative element $\Delta$

$$\Phi = \Phi(\Delta_A \boldsymbol{u}^a, \boldsymbol{u}^b - \boldsymbol{u}^c) \tag{4.7}$$

Eq (4.7) also holds for an arbitrary element $\Delta(\boldsymbol{z}) = \boldsymbol{z} + \Delta$, $\boldsymbol{z} \in \Lambda$ provided that arguments $\boldsymbol{u}^a$ are replaced by $\boldsymbol{u}^a(\boldsymbol{z}, t)$. Hence we arrive at Eqs (3.3) and (3.4) as the governing equations of a linear-elastic composite material in the finite element approximation.

## 5. Applications

The proposed simplicial model of plane problems for dynamics of linear elastic periodic media, represented by Eqs (3.3), (3.4) combined with Eqs (4.1) $\div$ (4.7), will be now applied to investigation of a plane wave propagating across the unbounded linear elastic medium with periodically spaced rigid prizmatic inclusions. The rectangular fragment of cross-section of the medium by an arbitrary plane $x_3$ = const is shown in Fig.1. The cross-sections of rigid inclusions on $0x_1x_2$ plane constitute equilateral triangles having mass densities $\rho_1, \rho_2$ per unit area. Hence, the representative element $\Delta$, bounded in Fig.1 by the bold line, is composed of two equilateral hexagons. The elastic medium is assumed to be homogeneous and isotropic with the mass density $\rho_0$ and Lame moduli $\lambda, \mu$. We shall investigate an oblique wave propagating in the direction normal to vector $\boldsymbol{d}_2$ with the displacements of rigid inclusions parallel to the vector $\boldsymbol{d}_1$ as shown in Fig.1.

To apply the model proposed in this contribution we shall restrict ourselves to the simplicial subdivision of $\Delta$ into 12 equilateral triangles (symplexes) which are shown in Fig.2. In the problem under consideration we can assume $N = 1$ and hence the finite differences will be denoted by

$$\begin{aligned} \Delta f &= \Delta_1 f(\boldsymbol{z}) \equiv f(\boldsymbol{z} + \boldsymbol{d}_1) - f(\boldsymbol{z}) \\ \overline{\Delta} f &= \overline{\Delta}_1 f(\boldsymbol{z}) \equiv f(\boldsymbol{z}) - f(\boldsymbol{z} - \boldsymbol{d}_1) \end{aligned}$$

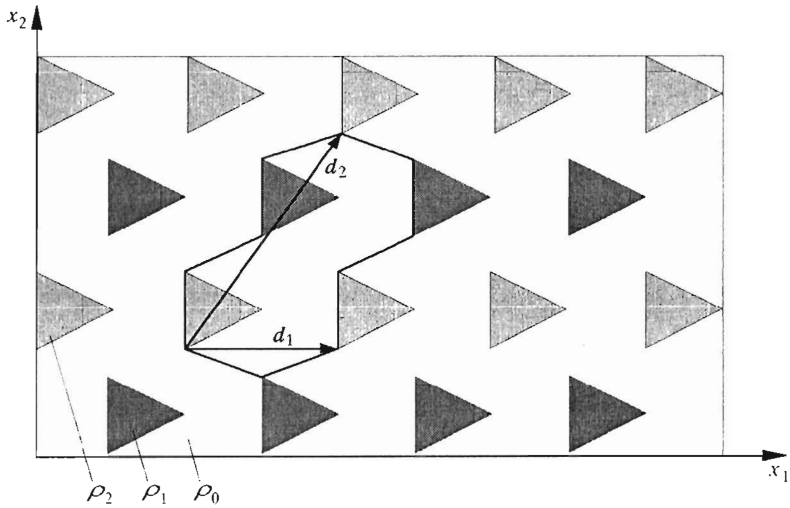


Fig. 1.

Let us denote by $u_1, u_2$ the displacements of rigid inclusions with mass densities $\rho_1, \rho_2$, respectively, and by $m_1, m_2$ the masses of two corresponding hexagons in $\Delta$, cf Fig.2. Let us also introduce new elasticity modulae

$$\alpha = \frac{\sqrt{3}}{12}(\lambda + 2\mu) \qquad \beta = \frac{\sqrt{3}}{4}\mu$$

Using Eqs (4.1) $\div$ (4.7), after rather tedious calculations, we obtain from Eqs (3.3), (3.4) the following system of finite difference equations for $u_1, u_2$

$$\begin{aligned} &8\alpha\overline{\Delta}\Delta u_1 + 2(\alpha+\beta)\overline{\Delta}\Delta u_2 + 8(\alpha+\beta)(u_2 - u_1) - m_1\ddot{u}_1 = 0 \\ &8\alpha\overline{\Delta}\Delta u_2 + 2(\alpha+\beta)\overline{\Delta}\Delta u_1 + 8(\alpha+\beta)(u_1 - u_2) - m_2\ddot{u}_2 = 0 \end{aligned} \tag{5.1}$$

Introducing the new unknows $U = \frac{1}{2}(u_1 + u_2)$, $V = \frac{1}{2}(u_1 - u_2)$, and denoting

$$\begin{aligned} \kappa &= 4(5\alpha + \beta) \qquad & \eta &= 4(3\alpha - \beta) \\ m &= m_1 + m_2 \qquad & \tilde{m} &= m_1 - m_2 \end{aligned}$$

we shall rewrite Eqs (5.1) in the simple form

$$\begin{aligned} &\kappa\overline{\Delta}\Delta U - m\ddot{U} - \tilde{m}\ddot{V} = 0 \\ &\eta\overline{\Delta}\Delta V - 4(\kappa - \eta)V - m\ddot{V} - \tilde{m}\ddot{U} = 0 \end{aligned} \tag{5.2}$$

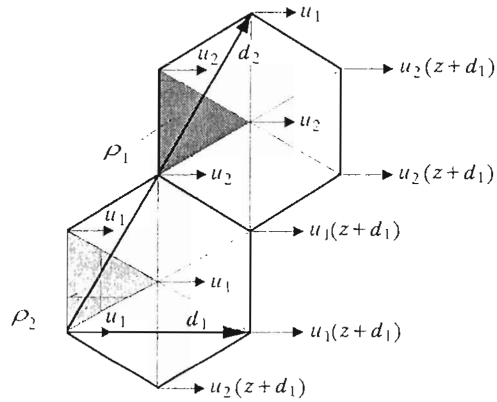


Fig. 2.

Setting $k = 2\pi d_1/L$ where $d_1 = |\boldsymbol{d}_1|$ and $L$ is the wavelength, and substituting

$$U = A_U \exp[\mathrm{i}(\omega t - nk)] \qquad V = A_V \exp[\mathrm{i}(\omega t - nk)] \qquad n = 0, \pm 1, \pm 2, \ldots$$

into Eqs (5.2) we obtain

$$\begin{aligned} &A_U[2\kappa(\cos k - 1) + m\omega^2] + \tilde{m}\omega^2 A_V = 0 \\ &A_V[2\eta(\cos k - 1) - 4(\kappa - \eta) + m\omega^2] + \tilde{m}\omega^2 A_U = 0 \end{aligned} \tag{5.3}$$

From the above system of equations for amplitudes $A_U, A_V$ we derive the following dispersion relation

$$m_1 m_2 \omega^4 - \Big[(\kappa+\eta)\sin^2\frac{k}{2} + \kappa - \eta\Big] m\omega^2 + 4\kappa \sin^2\frac{k}{2}\Big(\eta \sin^2\frac{k}{2} + \kappa - \eta\Big) = 0 \tag{5.4}$$

It can be proved that there exist two non-intersecting branches (lower and upper) of the above dispersion relation. For a special case $m_1 = m_2$ these branches are shown in Fig.3, where $A_1 = A_U + A_V$, $A_2 = A_U - A_V$ are vibration amplitudes related to the rigid inclusions shown in Fig.2. If

$$\frac{m_2}{m_1} < \frac{\kappa - \eta}{\kappa + \eta}$$

then there exist the stopping band, i.e. the maximum lower frequency $\omega_-$ for $k = \pm\pi$ is smaller than the minimum upper frequency $\omega_+$ for $k = 0$.

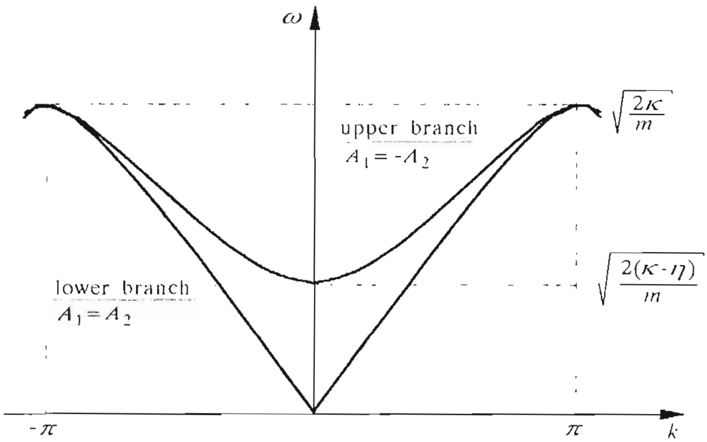


Fig. 3.

So far, the dispersion analysis was of a quite formal character. It has to be emphasized, however, that the simplicial subdivision of $\Delta$ into 12 equilateral triangles, each undergoing exclusively a uniform strain, has a physical sense only under condition $d_1 \ll L$, i.e., for the lengthwaves sufficiently large compared with the length dimension of cell $\Delta$ along the $x_1$-axis. In this case $k = 2\pi d_1/L$ is a small parametr, $k \ll 1$, and Eq (5.4) yields the following asymptotic formulae for the lower $\omega_-$ and upper $\omega_+$ free vibration frequencies

$$\begin{aligned} \omega_-^2 &= \frac{\kappa}{m} k^2 + o(k^4) \\ \omega_+^2 &= \frac{m}{m_1 m_2}(\kappa - \eta) + \Big[\frac{m}{4m_1 m_2}(\kappa + \eta) - \frac{\kappa}{m}\Big] k^2 + o(k^4) \end{aligned} \tag{5.5}$$

Setting $\cos k - 1 \cong -k^2/2$ in Eqs (5.3) and neglecting higher order terms in Eqs (5.5) it can be shown that for $\omega = \omega_-$ we obtain $A_V = 0$ and hence $A_1 = A_2$ while for $\omega = \omega_+$ we arrive at $A_U = 0$ and hence $A_1 = -A_2$.

The results similar to those obtained above hold also true for a plane wave propagating in the direction normal to the vector $\boldsymbol{d}_1$ with displacements of rigid inclusions in the direction of vector $\boldsymbol{d}_2$; in this case the dimensionless wave number is given by $k = 2\pi d_2/L$ with $d_2 = |\boldsymbol{d}_2|$.

## 6. Conclusions

The main conclusion of this contribution is that the periodic mass-point systems with a special distribution of ternary interactions which were specified in Section 3, can be applied to the analysis of plane dynamic problems for the linear-elastic periodic composites. The main advantage of the governing equations (3.3), (3.4) derived in Section 3 is their simple finite-difference form which is identical for every problem. Moreover, the above equations are rather general being formulated for an arbitrary simplicial periodic subdivision of the plane $E^2$. It follows that they also hold for an arbitrary decomposition of the representative composite element into finite elements provided that it implies the periodic simplicial subdivision of $E^2$. Hence Eqs (3.3), (3.4) can represent dynamic behaviour of composite materials with a required degree of accuracy. On the other hand, this requirement makes the formal structure of Eqs (3.3), (3.4) more involved due to a large number $n$ of functions $\boldsymbol{u}^a(\cdot,t)$, $a = 1,...,n$, defined on $\Lambda$. This is a main drawback in direct applications of the above equations to investigations of dynamic problems in composite materials. However, the form of these equations implies that they can be also interpreted as finite difference approximations of a certain generalized continuum medium with a large number of local degrees of freedom represented by smooth fields $\boldsymbol{u}^a(\cdot,t)$, $a = 1,...n$, defined on $E^2$. The problems related to different applications of Eqs (3.3), (3.4) which include also dynamics of bounded composite solids will be studied in forthcoming papers.

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## Model simplicjalny zagadnień dynamiki ośrodków periodycznych

Streszczenie

Celem pracy jest zaproponowanie modelu dyskretnego zagadnień dynamiki w liniowo sprężystych materiałach kompozytowych. Model ten stanowi klasa periodycznych układów punktów materialnych o złożonej strukturze i ternarnych oddziaływaniach. Otrzymane wyniki to nowe matematyczne narzędzie służące do analizy zagadnień propagacji fal w niejednorodnych ośrodkach periodycznych. Ogólne rozważania zilustrowano przykładem analizy propagacji fal w ośrodku jednorodnym ze sztywnymi inkluzjami.

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