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COMPARISON OF TWO THEORIES OF GEOMETRICALLY NONLINEAR SHELLS

ANNA SŁAWIANOWSKA

Institute of Fundamental Technological Research Polish Academy of Sciences

A comparison between two approaches to a problem of geometrically nonlinear shells theories has been performed.
The shell strain tensors and equilibrium equations obtained by Duszek [1 ÷ 3] and Schmidt and Weichert [4] were investigated and compared. It was shown that three of the theories derived by Duszek [1 ÷ 3] are particular cases of the theory proposed by Schmidt and Weichert [4].

Key words: shells, geometrical nonlinearity

1. Introduction

In this paper we shall compare the approach used by Duszek [1 ÷ 3] with that employed by Schmidt and Weichert [4] to the geometrically nonlinear problem of plastic shells.

Although the aims and methods of both approaches are different, it will be shown that in certain cases of shell deformation strain tensors assume the same form. Moreover, the equilibrium equations in the rate form can also be compared, and as we will see, they also coincide. Such a coincidence occurs in the case of moderately large displacements in the sense of [1 ÷ 3] and moderate rotations in the sense of [4]. Both in [4] and [1 ÷ 3] it is assumed that the strains are small, however, the displacements are finite.

The essential feature of these approaches is the fact that the authors desist from certain assumptions of the Kirchhoff-Love theory. As it is known they say that rectilinear material fibres orthogonal to the undeformed midsurface, remain rectilinear and orthogonal to the deformed midsurface and do not change their lengths.

Duszek [1 ÷ 3] deals with thin and rigid plastic shells only. In the paper by Schmidt and Weichert [4] the shell thickness is not required to be thin.

The geometrical relations for the shells considered in [1 ÷ 3] are classified depending on their initial shape and assumed deformation modes.

In [4] a mathematically consistent description of the deformation of elastic-plastic shells has been proposed. The rotations which are assumed to be moderately large, are the rotations of the normals to the midsurface only [4].

2. Assumptions and notations

When formulating equations of the shell theory, both Duszek [1 ÷ 3] and Schmidt and Weichert [4], use the Lagrangian description, i.e. all quantities are referred to the undeformed shell configuration. The state of deformation is then defined by the Green strain tensor **E** and the state of stress by the second Piola-Kirchhoff stress tensor **S**.

In [1 ÷ 3], [5] and [4] 3D problems for shells have been reduced to the corresponding 2D forms. Such a reduction is possible provided that the stress resultants, which are the integrals over the shell thickness of certain stress functions are introduced. To perform this reduction, the Green strain tensor **E** is decomposed into $E_{\Gamma\Delta}$, $E_{\Delta 3}$, E_{33}; $\Gamma, \Delta = 1,2$.

It will be useful here to make a comparison of the corresponding notations applied in [1 ÷ 3] and [4]. Below, Latin indices run over $1,2,3$ and Greek indices take values $1,2$.

Table 1

[1 ÷ 3]	Meaning	[4]
X^1, X^2, X^3	material curvilinear coordinates	$\Theta^1, \Theta^2, \Theta^3$
$\boldsymbol{U}$	shell displacement vector	$\boldsymbol{V} = (V_\alpha, V_3)$
$\{G_\Delta, G_3\}$	local base of the undeformed shell	$\{g_\alpha, g_3\}$
$G_A \cdot G_B = G_{AB}$	metric tensor of the undeformed shell space	$g_i \cdot g_j = g_{ij}$
A_{AB}	metric tensor of the misurface	$a_\alpha \cdot a_\beta = a_{\alpha\beta}$
$B_{\Gamma\Delta}$	second fundamental surface tensor of the undeformed shell surface	$b_{\alpha\beta}$
S^{KL}	second Piola-Kirchhoff stress tensor	s^{ij}
E_{KL}	Green strain tensor	E_{ij}
U_Γ, U_3	displacements referred to the base vectors of the midsurface	v_α, v_3

$[1 \div 3]$	Meaning	[4]
$h(2H)$	shell wall thickness	h
$2L$	length of the deformation wave of the shell	–
R	minimum radius the shell curvature	–
$\lambda_{\Delta\Gamma}$	extension tensor of the midsurface	$\overset{0}{E}_{\alpha\beta}$
$\kappa_{\Delta\Gamma}$	change of curvature tensor of midsurface of the shell	$\overset{1}{E}_{\alpha\beta}$
$N^{\Delta\Gamma}, Q^{\Delta}; M^{\Delta\Gamma}$	stress resultants of the shell: normal force, shear force, bending moment	$\overset{0}{L}{}^{ij}; \overset{1}{L}{}^{ij}$
V_{Θ}, W	tangential and normal displacements of a point on the middle surface	$\overset{0}{v}_{\alpha}, \overset{0}{v}_3$
β_{Γ}, β_3	inclination of the external normal to the reference surface), the normal strain distribution over the reference surface	$\overset{1}{v}_{\alpha}, \overset{1}{v}_3$

3. Approach applied by Duszek $[1 \div 3]$

In $[1 \div 3]$ several classes of nonlinear geometrical relations for thin rigid-plastic shells are listed. The classification was performed with regard to the initial shape of the shell (shallow, moderately shallow or deep one) and the deformation mode (displacements: small, moderately large or large ones).

In this manner, simpler formulas of the nonlinear geometrical relations are obtained. Both in $[1 \div 3]$ and in [4] it is assumed that the components of the displacement vector U_K are linear functions of X^3 – the coordinate normal to the shell midsurface

$$
\begin{aligned}
U_{\Gamma} &= V_{\Gamma}(X^{\Delta}) + X^3 \beta_{\Gamma}(X_{\Delta}) \\
U_3 &= W(X_{\Delta}) + X^3 \beta_3(X_{\Delta})
\end{aligned}
\tag{3.1}
$$

Here V_{Γ}, W are the tangential and normal displacements of a point on the midsurface, β_{Γ} stands for the inclination of the external normal to the midsurface and β_3 specifies the normal strain distribution over the reference surface.

The Green strain tensor can be written in the form

$$
\begin{aligned}
2E_{\Delta\Gamma} &= U_\Delta\|_\Gamma + U_\Gamma\|_\Delta + U_\Phi\|_\Delta U^\Phi\|_\Gamma + U_3\|_\Delta U^3\|_\Gamma \\
2E_{\Delta 3} &= U_3\|_\Delta + U_\Delta\|_3 + U_\Phi\|_3 U^\Phi\|_\Delta + U_3\|_3 U^3\|_\Delta \\
2E_{33} &= 2U_3\|_3 + U_\Phi\|_3 U^\Phi\|_3 + U_3\|_3 U^3\|_3
\end{aligned}
\tag{3.2}
$$

where $\|$ denotes covariant differentiation with respect to the metric of the undeformed shell space.

In papers [1 ÷ 3] only thin shells are examined, i.e. $h/R \ll 1$.

It is assumed that no volume changes take place during the plastic deformation. The incompressibility condition is expressed by $E_{33} = -E_{\Delta\Gamma}G^{\Delta\Gamma}$ (in the case of rigid-plastic material of the shell). The shear strains are taken as negligible, i.e. $E_{\Delta 3} = 0$.

For thin shells the covariant spatial derivatives of space vectors are expressed in terms of the covariant surface derivatives of the surface representation of these vectors as follows, [1 ÷ 3], [5]

$$
\begin{aligned}
V_\Delta\|_\Gamma &= V_\Delta|_\Gamma - B_{\Delta\Gamma}W \\
W\|_\Delta &= W,_\Delta + B^\Gamma_\Delta V_\Gamma \\
V_\Delta\|_3 &= V_{\Delta,3}
\end{aligned}
\tag{3.3}
$$

where $|$ denotes covariant differentiation with respect to the metric of the midsurface of the undeformed shell.

Making use of the kinematic assumption (3.1) and the rules (3.3), for the Green strain tensor (3.2) we get the formulas

$$
\begin{aligned}
E_{\Delta\Gamma} &= V_{(\Delta}|_{\Gamma)} - B_{\Delta\Gamma}W + \frac{1}{2}\Big(W|_\Delta W|_\Gamma + B^\Phi_\Delta B_{\Phi\Gamma}W^2 + V^\Phi|_{(\Delta}V_\Phi|_{\Gamma)} + \\
&+ \ldots\Big) + \ldots + X^3\Big\{\beta_{(\Delta}|_{\Gamma)} - B_{\Delta\Gamma}\beta_3 + V^\Phi|_{(\Delta}\beta_\Phi|_{\Gamma)} - B^\Phi_{(\Delta}\beta_\Phi|_{\Gamma)}W + \\
&+ B^\Phi_{(\Delta}B_{\Phi\Gamma)}W\beta_3 + \ldots\Big\} \\
E_{\Delta 3} &= \frac{1}{2}\Big(\beta_\Delta + W|_\Delta + B^\Phi_\Delta V_\Phi + V^\Phi|_\Delta\beta_\Phi + W|_\Delta\beta_3 - B^\Phi_\Delta\beta_\Phi W + \ldots\Big) + \\
&+ \frac{1}{2}X^3\Big\{\beta_3|_\Delta + B^\Phi_\Delta\beta_\Phi + \beta^\Phi|_\Delta\beta_\Phi + B^\Gamma_\Phi B^\Phi_\Delta\beta_\Gamma W - B^\Gamma_\Phi\beta_\Gamma V^\Phi|_\Delta + \ldots\Big\} \\
E_{33} &= \beta_3 + \frac{1}{2}\Big(\beta^\Delta\beta_\Delta + (\beta_3)^2\Big) - X^3\Big\{B^\Gamma_\Phi\beta_\Gamma\beta^\Phi\Big\}
\end{aligned}
\tag{3.4}
$$

Obviously we have [2]

$$E_{\Delta\Gamma} = \lambda_{\Delta\Gamma} + X^3\kappa_{\Delta\Gamma}$$
$$E_{\Delta 3} = \lambda_{\Delta 3} + X^3\kappa_{\Delta 3} \tag{3.5}$$
$$E_{33} = \lambda_{33} + X^3\kappa_{33}$$

An order of magnitude of the particular components in the above formulas depends upon the state of deformation and the geometry of the shell. To formulate approximate theories several detailed cases have been considered by Duszek [1 ÷ 3] for the following four quantities: L/R, h/R, W/R, V_Δ/R.

We shall now consider three representative models studied by Duszek [1 ÷ 3]:

(a) Shallow shells: moderately large W and small V_Δ; i.e.

$$\frac{W}{h} = 1 \qquad \frac{V_\Delta}{h} = \varepsilon \qquad \frac{h}{R} = \varepsilon^2 \qquad \frac{L}{R} = \varepsilon$$

where $\varepsilon^2 = O(h/R)$, $\varepsilon^2 \ll 1$ and

$$\lambda_{\Delta\Gamma} = V_{(\Delta|\Gamma)} - B_{\Delta\Gamma}W + \frac{1}{2}W|_\Delta W|_\Gamma$$
$$\kappa_{\Delta\Gamma} = \beta_{(\Delta|\Gamma)} = -W|_{\Delta\Gamma}$$

(b) Quasi-shallow shells: moderately large W and small V_Δ; i.e.

$$\frac{W}{h} = 1 \qquad \frac{V_\Delta}{h} = \varepsilon \qquad \frac{h}{R} = \varepsilon^2 \qquad \frac{L}{R} = 1$$

and

$$\lambda_{\Delta\Gamma} = V_{(\Delta|\Gamma)} - B_{\Delta\Gamma}W$$
$$\kappa_{\Delta\Gamma} = -W|_{\Delta\Gamma} - B^\Phi_\Delta|_\Gamma V_\Phi - B^\Phi_{(\Delta}V_\Phi|_{\Gamma)} + B_{\Delta\Gamma}\Big(V^\Phi|_\Phi - B^\Phi_\Phi W\Big)$$

(c) Deep shells: moderately large W and small V_Δ; i.e.

$$\frac{W}{h} = 1 \qquad \frac{V_\Delta}{h} = \varepsilon \qquad \frac{h}{R} = \varepsilon^2 \qquad \frac{L}{R} = \frac{1}{\varepsilon}$$

and

$$\lambda_{\Delta\Gamma} = V_{(\Delta|\Gamma)} - B_{\Delta\Gamma}W$$
$$\kappa_{\Delta\Gamma} = -W|_{\Delta\Gamma} - B_{\Delta\Gamma}B^\Phi_\Phi W$$

In order to obtain all above relations; i.e., the cases **(a)**, **(b)** and **(c)**, an evaluation of the order of magnitude of each term in the adequate expressions for $\lambda_{\Delta\Gamma}$ and $\kappa_{\Delta\Gamma}$ has been performed.

In the formula for $\lambda_{\Delta\Gamma}$ the terms smaller than, or equal to, ε^2 as compared with the greatest linear term are neglected while all the linear terms are retained.

As what concerns the change of curvature $\kappa_{\Delta\Gamma}$ for nonlinear theory, the linear expression has been complemented by nonlinear terms of the same order of magnitude as the greatest linear term. Moreover the linear terms which are sufficiently small are neglected.

The six unknowns of the problem: β_Γ, β_3, W, V_Γ have thus been reduced to the following three ones: W and V_Γ. To derive these models the assumptions $E_{\Delta 3} = 0$ and $E_{33} = -E_{\Delta\Gamma}G^{\Delta\Gamma}$ were accepted.

4. Approach used by Schmidt and Weichert [4]

After Schmidt and Weichert [4] the Green strain tensor E_{ij} may be expressed by the formula

$$E_{ij} = \eta_{ij} + \frac{1}{2}\Omega_{ri}\Omega^r_j + \frac{1}{2}(\eta_{ri}\Omega^r_j + \eta_{rj}\Omega^r_i) + \frac{1}{2}\eta_{ri}\eta^r_j \tag{4.1}$$

where η_{ij} are components of the linearized strain tensor and Ω_{ij} are the linearized rotations given by

$$\eta_{ij} = \frac{1}{2}\Big(V_i\|_j + V_j\|_i\Big) \qquad\qquad \Omega_{ij} = \frac{1}{2}\Big(V_i\|_j - V_j\|_i\Big) \tag{4.2}$$

The assumption of small strains means that

$$E_{ij} = O(\vartheta^2) \qquad\qquad \vartheta^2 \ll 1 \tag{4.3}$$

In [4] the rotations are restricted by

$$\Omega_{\alpha\beta} = O(\vartheta^2) \qquad\qquad \Omega_{\alpha 3} = O(\vartheta) \tag{4.4}$$

Relation $(4.4)_2$ means that *rotations of the normals to the midsurface are moderately large*; consequently

$$\eta_{ij} = O(\vartheta^2) \tag{4.5}$$

Retaining in Eq (4.1) the terms of the order not exceeding $O(\Theta^3)$ only the following relations are obtained

$$
\begin{aligned}
E_{\alpha\beta} &= \eta_{\alpha\beta} + \frac{1}{2}\Omega_{3\alpha}\Omega_{3\beta} + \frac{1}{2}(\eta_{3\alpha}\Omega_{3\beta} + \eta_{3\beta}\Omega_{3\alpha}) + O(\vartheta^4) \\
E_{\alpha 3} &= \eta_{\alpha 3} + \frac{1}{2}\Omega_{\lambda 3}\Omega^{\lambda}_{\alpha} + \frac{1}{2}(\eta_{\lambda\alpha}\Omega^{\lambda}_{3} + \eta_{33}\Omega_{3\alpha}) + O(\vartheta^4) \\
E_{33} &= \eta_{33} + \frac{1}{2}\Omega_{\lambda 3}\Omega^{\lambda}_{3} + \eta_{\lambda 3}\Omega^{\lambda}_{3} + O(\vartheta^4)
\end{aligned}
\tag{4.6}
$$

The kinematical hypothesis assumed by Schmidt and Weichert [4] is given by

$$
v_\alpha = \overset{0}{v}_\alpha + \Theta^3 \overset{1}{v}_\alpha \qquad\qquad v_3 = \overset{0}{v}_3 + \Theta^3 \overset{1}{v}_3 \tag{4.7}
$$

Eqs (4.7) mean that the displacements, referred to the base vectors of the midsurface are distributed linearly across the shell thickness.

Making standard calculations and taking into account the order of magnitude of particular terms occuring in Eqs (4.6), the following 2D strain-displacements relations for shells undergoing small strains and moderate rotations of the normal are obtained

$$
E_{ij}(\Theta^1, \Theta^2, \Theta^3) = \sum_{n=0}^{2} (\Theta^3)^n \overset{n}{E}_{ij}(\Theta^1, \Theta^2) \tag{4.8}
$$

where

$$
\begin{aligned}
\overset{0}{E}_{\alpha\beta} &= \overset{0}{\Theta}_{\alpha\beta} + \frac{1}{2} \overset{0}{\varphi}_\alpha \overset{0}{\varphi}_\beta + O(\vartheta^4) \\
\overset{1}{E}_{\alpha\beta} &= \overset{1}{\Theta}_{\alpha\beta} - \frac{1}{2}\Big(b^\lambda_\alpha \overset{0}{\varphi}_{\lambda\beta} + b^\lambda_\beta \overset{0}{\varphi}_{\lambda\alpha}\Big) + \frac{1}{2}\Big(\overset{0}{\varphi}_\alpha \overset{1}{\varphi}_\beta + \overset{0}{\varphi}_\beta \overset{1}{\varphi}_\alpha\Big) + O\Big(\frac{\vartheta^4}{h}\Big) \\
\overset{2}{E}_{\alpha\beta} &= -\frac{1}{2}\Big(b^\lambda_\alpha \overset{1}{\varphi}_{\lambda\beta} + b^\lambda_\beta \overset{1}{\varphi}_{\lambda\alpha}\Big) + \frac{1}{2} \overset{1}{\varphi}_\alpha \overset{1}{\varphi}_\beta + O\Big(\frac{\vartheta^4}{h^2}\Big)
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\overset{0}{E}_{\alpha 3} &= \frac{1}{2}\Big(\overset{0}{\varphi}_\alpha + \overset{1}{v}_\alpha\Big) + \frac{1}{2} \overset{1}{v}^\lambda \overset{0}{\varphi}_{\lambda\alpha} + \ldots + O(\vartheta^4) \\
\overset{1}{E}_{\alpha 3} &= \frac{1}{2} \overset{1}{v}_3 |_\alpha + \frac{1}{2} \overset{1}{v}^\lambda \overset{1}{v}_\lambda |_\alpha + \ldots + O\Big(\frac{\vartheta^4}{h}\Big) \\
\overset{0}{E}_{33} &= \overset{1}{v}_3 + \frac{1}{2} \overset{1}{v}^\lambda \overset{1}{v}_\lambda + O(\vartheta^4)
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
&\overset{n}{E}_{\alpha\beta} \equiv 0 && \text{for } n \geq 3 \\
&h^n \overset{n}{E}_{\alpha 3} = O(\vartheta^4) && \text{for } n \geq 2 \\
&h^n \overset{n}{E}_{33} = O(\vartheta^4) && \text{for } n \geq 1
\end{aligned}
\tag{4.11}
$$

and for $n = 0, 1$

$$\begin{aligned}
\overset{n}{\Theta}_{\alpha\beta} &= \frac{1}{2}\left(\overset{n}{v}_{\alpha}\,|_{\beta} + \overset{n}{v}_{\beta}\,|_{\alpha}\right) - b_{\alpha\beta}\,\overset{n}{v}_{3} \\
\overset{n}{\varphi}_{\alpha\beta} &= \overset{n}{v}_{\alpha}\,|_{\beta} - b_{\alpha\beta}\,\overset{n}{v}_{3} \\
\overset{n}{\varphi}_{\alpha} &= \overset{n}{v}_{3}\,|_{\alpha} + b_{\alpha}^{\lambda}\,\overset{n}{v}_{\lambda}
\end{aligned} \tag{4.12}$$

5. Comparison of the geometrical relations derived by Duszek [1 ÷ 3] and Schmidt and Weichert [4]

As we have noticed before there are common points in the papers by Duszek [1÷3] and by Schmidt and Weichert [4]. Obviously, some assumptions coincide.

Now we shall demonstrate that under suitable conditions imposed on the theory of Schmidt and Weichert [4], the results obtained by Duszek [1 ÷ 3] are recovered. Let us first list these assumptions:

(A$_1$) $E_{\Delta 3} = 0$ (transverse shear strains)

(A$_2$) Plastic incompressibility

(A$_3$) $\left|W|_{\Gamma}\right| = O\left(\frac{W}{L}\right)$, $\left|V_{\Delta}|_{\Gamma}\right| = O\left(\frac{V_{\Delta}}{L}\right)$

(A$_4$) $B_{\Gamma}^{\Delta} = O\left(\frac{1}{R}\right)$

(A$_5$) $\beta_3 = 0 \quad \rightarrow \quad U_3 = W(X^{\Delta})$

(A$_6$) $\sigma^{33} = 0$ (Cauchy normal stress).

The assumptions **(A$_5$)** and **(A$_6$)** will be applied in Section 7.

There are six quantities describing the mathematical problem in [4]: $\overset{0}{v}_{\alpha}$, $\overset{1}{v}_{\alpha}$, $\overset{0}{v}_{3}$, $\overset{1}{v}_{3}$ and corresponding to them in [1 ÷ 3] six quantities: V_{Γ}, β_{Γ}, W, β_3.

Let the notation $\Leftrightarrow$ refer to the corresponding quantities of both approaches. We pass to proving that

$$\overset{0}{E}_{\alpha\beta} \Leftrightarrow \lambda_{\alpha\beta} \qquad\qquad \overset{1}{E}_{\alpha\beta} \Leftrightarrow \kappa_{\alpha\beta} \tag{5.1}$$

We have (cf [4])

$$\overset{0}{E}_{\alpha\beta} = \overset{0}{\Theta}_{\alpha\beta} + \frac{1}{2}\,\overset{0}{\varphi}_{\alpha}\overset{0}{\varphi}_{\beta} + O(\vartheta^4)$$

where $\overset{0}{\Theta}_{\alpha\beta}$, $\overset{0}{\varphi}_{\alpha}$ are defined by Eqs (4.13).

It is easy to see that

$$\overset{0}{\Theta}_{\alpha\beta} \Leftrightarrow V_{\Delta}|_{\Gamma} - B_{\Delta\Gamma} W \tag{5.2}$$

which is the linear term of the strain tensor.

The nonlinear term $\frac{1}{2}\overset{0}{\varphi}_{\alpha}\overset{0}{\varphi}_{\beta}$ *of* $\overset{0}{E}_{\alpha\beta}$ *is to be compared with* $\frac{1}{2}W|_{\Delta}W|_{\Gamma}$ of Duszek [1 ÷ 3]. Then we have

$$\begin{aligned} \overset{0}{\varphi}_{\alpha}\overset{0}{\varphi}_{\beta} \;\Leftrightarrow\; & (W|_{\alpha} + B^{\lambda}_{\alpha}V_{\lambda})(W|_{\beta} + B^{\delta}_{\beta}V_{\delta}) = \\ = {} & W|_{\alpha}W|_{\beta} + B^{\lambda}_{\alpha}V_{\lambda}W|_{\beta} + B^{\delta}_{\beta}V_{\delta}W|_{\alpha} + B^{\lambda}_{\alpha}V_{\lambda}B^{\delta}_{\beta}V_{\delta} \end{aligned} \tag{5.3}$$

Now we shall examine the terms appearing in Eq (5.3) for the three cases of the "order of magnitude" specified by **(a)**, **(b)**, **(c)**, see Section 3.

All components of $\lambda_{\alpha\beta}$ which are smaller than ε^2 are omitted. A corresponding simplification has to be performed within the theory developed by Schmidt and Weichert [4]. Accordingly, we have to impose

$$\frac{W}{h} = O(1) \qquad \frac{V_{\Delta}}{h} = O(\varepsilon) \qquad \frac{h}{R} = O(\varepsilon^2) \tag{5.4}$$

and

$$\frac{L}{R} = O(\varepsilon) \qquad \text{or} \qquad \frac{L}{R} = O(1) \qquad \text{or} \qquad \frac{L}{R} = O\Big(\frac{1}{\varepsilon}\Big)$$

Then

$$B^{\lambda}_{\alpha}V_{\lambda} = O\Big(\frac{1}{R}\Big)h\varepsilon = O\Big(\frac{\varepsilon^2}{h}\varepsilon h\Big) = O(\varepsilon^3)$$

see also assum'tion $(\mathbf{A}_4)$.

Thus we have to omit the second, third and fourth terms, respectively, on the right hand side of Eq (5.3). In this way only the first term is to be retained, i.e. $W|_{\alpha}W|_{\beta}$. Let us estimate this remaining term. We have

$$\Big|W|_{\alpha}\Big| = O\Big(\frac{W}{L}\Big) = O\Big(\frac{h}{L}\Big)$$

The following three cases are possible:

$$\begin{aligned} &(\text{a}) \quad L = R\varepsilon = \frac{h}{\varepsilon} && \Big|W|_{\alpha}\Big| = O(\varepsilon) \\ &(\text{b}) \quad L = R = \frac{h}{\varepsilon^2} && \Big|W|_{\alpha}\Big| = O(\varepsilon^2) \\ &(\text{c}) \quad L = \frac{R}{\varepsilon} = \frac{h}{\varepsilon^3} && \Big|W|_{\alpha}\Big| = O(\varepsilon^3) \end{aligned} \tag{5.5}$$

From Eqs (5.5) we infer that in the case of moderate deflections of shells, i.e. for $w \approx h$, the nonlinear component of $\overset{0}{E}_{\alpha\beta} \Leftrightarrow \lambda_{\alpha\beta}$, should be taken into account for thin shallow shells only, i.e. in the case **(a)**. And so it is formulated in [1 ÷ 3]. So, in general we have $\overset{0}{E}_{\alpha\beta} \Leftrightarrow \lambda_{\alpha\beta}$.

In the cases **(b)** and **(c)** the term $\frac{1}{2}W|_\alpha W|_\beta$ has to be neglected.

Now we may pass to the comparison of the curvature tensor $\kappa_{\Delta\Gamma}$ used by Duszek [1 ÷ 3] with its corresponding quantity $\overset{1}{E}_{\alpha\beta}$ applied by Schmidt and Weichert [4]. In accordance with [1 ÷ 3] we make two assumptions

$$E_{\alpha 3} = 0 \qquad -\frac{h}{2} \leq \Theta^3 \leq \frac{h}{2} \tag{5.6}$$

$$E_{33} = -E_{\alpha\beta}g^{\alpha\beta} \qquad -\frac{h}{2} \leq \Theta^3 \leq \frac{h}{2} \quad \text{(plastic incompressibility)} \tag{5.7}$$

From Eqs (5.6) and (5.7) for $\Theta^3 = 0$ we have

$$\overset{0}{E}_{\alpha 3} = 0 \tag{5.8}$$

$$\overset{0}{E}_{33} = -\overset{0}{E}_{\alpha\beta}\, g^{\alpha\beta} \tag{5.9}$$

Following Schmidt and Weichert [4] we write

$$\overset{0}{E}_{\alpha 3} = \frac{1}{2}(\overset{0}{\varphi}_\alpha + \overset{1}{v}_\alpha) + \frac{1}{2}\, \overset{1}{v}{}^\lambda\, \overset{0}{\varphi}_{\alpha\lambda} + \ldots \tag{5.10}$$

Taking into account the linear part of Eq (5.10) we get

$$\overset{0}{\varphi}_\alpha + \overset{1}{v}_\alpha = 0 \quad \rightarrow \quad \overset{1}{v}_\alpha = -\overset{0}{\varphi}_\alpha = -\overset{0}{v}_3\,|_\alpha - b^\lambda_\alpha\, \overset{0}{v}_\lambda \tag{5.11}$$

Now we shall consider the three cases of deformation, specified as cases **(a)**, **(b)** and **(c)**, (see Sections 3 and 5). For each of this cases we have

$$\textbf{(a)} \qquad \overset{1}{v}_\alpha \Leftrightarrow \beta_\alpha = -W|_\alpha \qquad\qquad W|_\alpha = O(\varepsilon) \tag{5.12}$$

$$\textbf{(b)} \qquad \overset{1}{v}_\alpha \Leftrightarrow \beta_\alpha = -W|_\alpha - B^\lambda_\alpha V_\lambda \qquad\qquad W|_\alpha = O(\varepsilon^2) \tag{5.13}$$

$$\textbf{(c)} \qquad \overset{1}{v}_\alpha \Leftrightarrow \beta_\alpha = -W|_\alpha - B^\lambda_\alpha V_\lambda \qquad\qquad W|_\alpha = O(\varepsilon^3) \tag{5.14}$$

Let us pass to the determination of $\overset{1}{v}_3 \Leftrightarrow \beta_3$. From Eq (5.9) we get

$$\overset{0}{E}_{33} = \overset{1}{v}_3 + \frac{1}{2}\, \overset{1}{v}{}^\lambda\, \overset{1}{v}_\lambda = -\overset{0}{E}_{\alpha\beta}\, g^{\alpha\beta} = -\overset{0}{\Theta}_{\alpha\beta}\, g^{\alpha\beta} - \frac{1}{2}\, \overset{0}{\varphi}_\alpha \overset{0}{\varphi}_\beta\, g^{\alpha\beta} + \ldots \tag{5.15}$$

Similarly as $\overset{1}{v}_\alpha \Leftrightarrow \beta_\alpha$ the quantity $\overset{1}{v}_3 = \overset{1}{v}_3\,(\overset{0}{v}_\lambda, \overset{0}{v}_3)$ should also be estimated for our three cases studied. We have to evaluate $V_\Delta|_\Gamma$ in the cases **(a)**, **(b)** and **(c)**.

$$
\begin{aligned}
&(\mathrm{a}) \quad V_\Delta|_\Gamma = O\Big(\frac{V_\Delta}{L}\Big) = O(\varepsilon^2)\\
&(\mathrm{b}) \quad V_\Delta|_\Gamma = O\Big(\frac{V_\Delta}{L}\Big) = O(\varepsilon^3)\\
&(\mathrm{c}) \quad V_\Delta|_\Gamma = O\Big(\frac{V_\Delta}{L}\Big) = O(\varepsilon^4)
\end{aligned}
\tag{5.16}
$$

We are going to examine Eq (5.15) in each of three cases. In the case **(a)** we have

$$
\overset{1}{v}_3 + \frac{1}{2}\,\overset{0}{v}_3\,|_\lambda\,\overset{0}{v}_3\,|^\lambda = -\,\overset{0}{v}_{(\alpha}\,|_{\beta)} g^{\alpha\beta} + b_{\alpha\beta}\,\overset{0}{v}_3\, g^{\alpha\beta} - \frac{1}{2}\,\overset{0}{v}_3\,|_\alpha\,\overset{0}{v}_3\,|_\beta g^{\alpha\beta} + \ldots \tag{5.17}
$$

Using the notation of Duszek [1 ÷ 3], making an obvious contraction we get

$$
\beta_3 = -V^\Phi|_\Phi + B^\Phi_\Phi W - W|_\Phi W|^\Phi \tag{5.18}
$$

Now, we can estimate $\beta_3 \Leftrightarrow \overset{1}{v}_3$ in Eq (5.18)

$$
\overset{1}{v}_3 = -\underbrace{\overset{0}{v}{}^\lambda|_\lambda}_{\approx O(\varepsilon^2)} + \underbrace{b^\lambda_\lambda\,\overset{0}{v}_3}_{\approx O(\varepsilon^2)} - \underbrace{\overset{0}{v}_3\,|_\lambda\,\overset{0}{v}_3\,|^\lambda}_{\approx O(\varepsilon^2)} \tag{5.19}
$$

According to the argumentation put forward at the and of Section 3, we neglect all three components of $\overset{1}{v}_3$ in the above expression. Finally for the case **(a)** we arrive at

$$
\overset{1}{E}_{\alpha\beta} \Leftrightarrow \kappa_{\alpha\beta} \Leftrightarrow \frac{1}{2}(\overset{1}{v}_\alpha\,|_\beta + \overset{1}{v}_\beta\,|_\alpha) \Leftrightarrow -\frac{1}{2}(W|_{\alpha\beta} + W|_{\beta\alpha}) = -W|_{\alpha\beta} \tag{5.20}
$$

Let us pass to the cases **(b)** and **(c)**.

From formulas (5.13), (5.14) and (5.17) we can derive and estimate all terms of $\overset{1}{v}_3$ $(\overset{1}{v}_3 \Leftrightarrow \beta_3)$. Let us first calculate $\beta_{(\alpha|\beta)}$ for the cases **(b)** and **(c)**. Not specifying yet the order of magnitude for both cases we may write

$$
\begin{aligned}
\beta_{(\alpha|\beta)} &= \frac{1}{2}(\beta_\alpha|_\beta + \beta_\beta|_\alpha) =\\
&= \frac{1}{2}(-W|_{\alpha\beta} - B^\lambda_\alpha|_\beta V_\lambda - B^\lambda_\alpha V_\lambda|_\beta - W|_{\beta\alpha} - B^\lambda_\beta|_\alpha V_\lambda - B^\lambda_\beta V_\lambda|_\alpha) =\\
&= -\frac{1}{2}(W|_{\alpha\beta} + W|_{\beta\alpha}) - \frac{1}{2}V_\lambda(B^\lambda_\alpha|_\beta + B^\lambda_\beta|_\alpha) - \frac{1}{2}(B^\lambda_\alpha V_\lambda|_\beta + B^\lambda_\beta V_\lambda|_\alpha) =\\
&= -W|_{\alpha\beta} - B^\lambda_\alpha|_\beta V_\lambda - B^\lambda_{(\alpha} V_\lambda|_{\beta)}
\end{aligned}
\tag{5.21}
$$

For the cases (**b**) and (**c**) the condition of plastic incompressibility expresses as follows

$$\overset{1}{v}_3 + \frac{1}{2}\, \overset{0}{v}_3 \,|^{\lambda}\, \overset{0}{v}_3 \,|_{\lambda} = -\, \overset{0}{v}_{(\alpha} \,|_{\beta)} g^{\alpha\beta} + b_{\alpha\beta}\, \overset{0}{v}_3\, g^{\alpha\beta} \tag{5.22}$$

as from the expression for $\overset{1}{v}{}^{\lambda}\, \overset{1}{v}_{\lambda}$ (see Eqs (5.13) and (5.14)) we take into account one member: $\overset{0}{v}_3 \,|^{\lambda}\, \overset{0}{v}_3 \,|_{\lambda}$ only, because three further ones are very high order of magnitude. From Eq (5.22) we obtain

$$\overset{1}{v}_3 = -\, \overset{0}{v}_{\alpha} \,|_{\alpha} + b^{\alpha}_{\alpha}\, \overset{0}{v}_3 - \frac{1}{2}\, \overset{0}{v}_3 \,|^{\alpha}\, \overset{0}{v}_3 \,|_{\alpha} \tag{5.23}$$

or in the notation used by Duszek [1 ÷ 3]

$$\beta_3 = -V^{\Gamma}|_{\Gamma} + B^{\Gamma}_{\Gamma} W - \frac{1}{2} W|^{\Gamma} W|_{\Gamma}$$

Now, one has to distinguish between the cases (**b**) and (**c**). The conditions concerning the order of magnitude from the end of Section 3 are still compulsory.

The case (**b**)

$$\begin{aligned} \overset{0}{v}_3 \,|^{\varphi}\, \overset{0}{v}_3 \,|_{\varphi} &= O(\varepsilon^4) \\ \overset{0}{v}{}^{\varphi}|_{\varphi} &= O(\varepsilon^3) \\ b^{\varphi}_{\varphi}\, \overset{0}{v}_3 &= O(\varepsilon^2) \end{aligned} \tag{5.24}$$

Omitting the last term of Eq (5.23) we get the formula for the change of curvature

$$\begin{aligned} \overset{1}{E}_{\alpha\beta} &= \Theta^{1}_{\alpha\beta} = v^{1}_{(\alpha}|_{\beta)} - b_{\alpha\beta}\, \overset{1}{v}_3 = \\ &= \overset{0}{v}_3 \,|_{\alpha\beta} - b^{\lambda}_{\alpha}|_{\beta}\, \overset{0}{v}_{\lambda} - b^{\lambda}_{(\alpha}\, \overset{0}{v}_{\lambda} \,|_{\beta)} - b_{\alpha\beta}(\overset{0}{v}{}^{\alpha}|_{\alpha} - b^{\alpha}_{\alpha}\, \overset{0}{v}_3) \end{aligned} \tag{5.25}$$

The case (**c**)

$$\begin{aligned} \overset{0}{v}_3 \,|^{\alpha}\, \overset{0}{v}_3 \,|_{\alpha} &= O(\varepsilon^6) \\ \overset{0}{v}{}^{\varphi}|_{\varphi} &= O(\varepsilon^4) \\ b^{\varphi}_{\varphi}\, \overset{0}{v}_3 &= O(\varepsilon^2) \end{aligned} \tag{5.26}$$

(see Eqs (5.5) and (5.16)). Thus we can neglect two terms of Eq (5.23), hence

$$\beta_3 \Leftrightarrow \overset{1}{v}_3 = b^{\alpha}_{\alpha} \overset{0}{v}_3 \tag{5.27}$$

Further, because of the order of magnitude, we can omit two terms of Eq (5.21) and obtain then

$$\kappa_{\alpha\beta} \Leftrightarrow \overset{1}{E}_{\alpha\beta} = \overset{1}{\Theta}_{\alpha\beta} = - \overset{0}{v}_3 |_{\alpha\beta} - b_{\alpha\beta} b^{\varphi}_{\varphi} \overset{0}{v}_3 \tag{5.28}$$

In all three cases **(a)**, **(b)** and **(c)** the higher order terms of $\overset{1}{E}_{\alpha\beta}$ have to be neglected.

6. Equilibrium equations (cf Duszek [$1 \div 3$])

In contrast to the kinematics, where three models of shells could be compared, the equilibrium equations will be examined only for the case **(a)**. Such a limitation is justified by the fact that in [$1 \div 3$] for the cases **(b)** and **(c)** the equilibrium equations have not been discussed.

The stress resultants for thin shells ($\mu^{\Gamma}_{\Delta} \simeq \delta^{\Gamma}_{\Delta}$, μ^{Γ}_{Δ}-shifter) are defined as follows

$$\begin{aligned} &\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} S^{\Delta\Gamma}\, dX^3 = N^{\Delta\Gamma} \qquad \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} X^3 S^{\Delta\Gamma}\, dX^3 = M^{\Delta\Gamma} \\ &\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} S^{\Delta 3}\, dX^3 = Q^{\Delta} \end{aligned} \tag{6.1}$$

In the case **(a)** (shallow shells undergoing moderately large deflections and small strains) the equilibrium equations in the rate form are

$$\begin{aligned} &\dot{N}^{\Delta\Gamma}|_{\Gamma} - B^{\Delta}_{\Gamma}\dot{Q}^{\Gamma} + (\dot{\beta}^{\Delta}Q^{\Gamma})|_{\Gamma} + (\beta^{\Delta}\dot{Q}^{\Gamma})|_{\Gamma} + \dot{P}^{\Delta} = 0 \\ &B_{\Delta\Gamma}\dot{N}^{\Delta\Gamma} + \dot{Q}^{\Gamma}|_{\Gamma} + (\dot{W}|_{\Delta}N^{\Delta\Gamma})|_{\Gamma} + (W|_{\Gamma}\dot{N}^{\Delta\Gamma})|_{\Gamma} + \dot{P} = 0 \\ &\dot{M}^{\Delta\Gamma}|_{\Gamma} = \dot{Q}^{\Delta} \end{aligned} \tag{6.2}$$

where

$$\beta_{\Delta} = -W|_{\Delta} \tag{6.3}$$

and

$$P^{\Gamma} = S^{3\Gamma}\Big|_{-h/2}^{+h/2} \qquad\qquad P = (S^{33} + W|_{\Delta} S^{3\Delta})\Big|_{-h/2}^{+h/2} \tag{6.4}$$

are the external loads applied to the mid-surface of the shell.

A superposed dot denotes the derivation with respect to time.

7. Equilibrium equations (cf Schmidt and Weichert [4])

To compare both approaches let us start by recalling the general form of the rate equilibrium equations derived by Schmidt and Weichert [4].

From the variational principle $\delta I(\dot{v}) = 0$, where $\boldsymbol{v} = (\overset{0}{v}_\alpha, \overset{0}{v}_3, \overset{1}{v}_\alpha, \overset{1}{v}_3)$ and the variational functional $I(\dot{v})$ in its 2D form given in [4], the six nonlinear rate equilibrium equations were obtained. Neglecting body forces we get the following set of equilibrium equations

$$\begin{aligned}
&\delta\,\overset{0}{\dot{v}}_\alpha : \quad \overset{0}{\dot{S}}{}^{\alpha\beta}|_\beta - b^\alpha_\beta\,\overset{0}{\dot{S}}{}^{3\beta} + \overset{0}{\dot{p}}{}^\alpha = 0\\
&\delta\,\overset{0}{\dot{v}}_3 : \quad \overset{0}{\dot{S}}{}^{3\beta}|_\beta + b_{\alpha\beta}\,\overset{0}{\dot{S}}{}^{\alpha\beta} + \overset{0}{\dot{p}}{}^3 = 0\\
&\delta\,\overset{1}{\dot{v}}_\alpha : \quad \overset{1}{\dot{S}}{}^{\alpha\beta}|_\beta - b^\alpha_\beta\,\overset{1}{\dot{\hat{S}}}{}^{3\beta} + \overset{1}{\dot{R}}{}^{\alpha 3} + \overset{1}{\dot{p}}{}^\alpha = 0\\
&\delta\,\overset{1}{\dot{v}}_3 : \quad \overset{1}{\dot{S}}{}^{3\beta}|_\beta + b_{\alpha\beta}\,\overset{1}{\dot{\hat{S}}}{}^{\alpha\beta} + \overset{1}{\dot{R}}{}^{33} + \overset{1}{\dot{p}}{}^3 = 0
\end{aligned} \tag{7.1}$$

where (see Eqs (4.13))

$$\begin{aligned}
\overset{0}{\dot{S}}{}^{\alpha\beta} &= \overset{0}{\dot{L}}{}^{\alpha\beta} - b^\alpha_\lambda\,\overset{1}{\dot{L}}{}^{\lambda\beta} + \overset{1}{\dot{v}}{}^\alpha\,\overset{0}{\dot{L}}{}^{\beta 3} + \overset{1}{\dot{v}}{}^\alpha\,\overset{0}{\dot{L}}{}^{\beta 3}\\
\overset{0}{\dot{S}}{}^{3\beta} &= \overset{0}{\dot{L}}{}^{\beta 3} + \overset{0}{\varphi}_\lambda \overset{0}{\dot{L}}{}^{\lambda\beta} + \overset{0}{\dot{\varphi}}_\lambda \overset{0}{L}{}^{\lambda\beta} + \overset{1}{\varphi}_\lambda \overset{1}{\dot{L}}{}^{\lambda\beta} + \overset{1}{\dot{\varphi}}_\lambda \overset{1}{L}{}^{\lambda\beta}\\
\overset{1}{\dot{S}}{}^{\alpha\beta} &= \overset{1}{\dot{L}}{}^{\alpha\beta} + (\overset{1}{v}{}^\alpha\,\overset{1}{L}{}^{\beta 3})^{\cdot} - b^\alpha_\lambda\,\overset{2}{\dot{L}}{}^{\lambda\beta}\\
\overset{1}{\dot{\hat{S}}}{}^{\alpha\beta} &= \overset{1}{\dot{L}}{}^{\alpha\beta} - b^\alpha_\lambda\,\overset{2}{\dot{L}}{}^{\lambda\beta}\\
\overset{1}{\dot{S}}{}^{3\beta} &= \overset{1}{\dot{L}}{}^{\beta 3} + (\overset{0}{\varphi}_\lambda \overset{1}{L}{}^{\lambda\beta})^{\cdot} + (\overset{1}{\varphi}_\lambda \overset{2}{L}{}^{\lambda\beta})^{\cdot} - \frac{1}{2}(\overset{0}{\varphi}{}^\beta_\lambda\,\overset{1}{L}{}^{\lambda 3})^{\cdot}\\
\overset{1}{\dot{\hat{S}}}{}^{3\beta} &= \overset{0}{\varphi}_\lambda \overset{1}{\dot{L}}{}^{\lambda\beta} + \overset{0}{\dot{\varphi}}_\lambda \overset{1}{L}{}^{\lambda\beta} + (\overset{1}{\varphi}_\lambda \overset{2}{L}{}^{\lambda\beta})^{\cdot}
\end{aligned} \tag{7.2}$$

$$\overset{1}{R}{}^{\alpha 3} = -\,\dot{\overset{0}{L}}{}^{\alpha 3} + (\overset{0}{\varphi}{}^{\alpha}_{\lambda}\,\overset{0}{L}{}^{\lambda 3})^{\boldsymbol{\cdot}} + (\overset{1}{v}{}^{\alpha}\,\overset{0}{L}{}^{33})^{\boldsymbol{\cdot}} + (\overset{1}{v}{}^{\alpha}|_{\lambda}\,\overset{1}{L}{}^{\lambda 3})^{\boldsymbol{\cdot}}$$

$$\overset{1}{R}{}^{33} = -\,\dot{\overset{0}{L}}{}^{33} - \frac{1}{2}\Big((\overset{1}{v}_{\alpha} - \overset{0}{\varphi}_{\alpha})\,\overset{0}{L}{}^{\alpha 3}\Big)^{\boldsymbol{\cdot}}$$

The nth order stress resultants $\overset{n}{L}{}^{ij}$ are here as follows

$$\overset{n}{L}{}^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\Theta^3)^n \mu s^{ij}\, d\Theta^3$$
$$\mu = \sqrt{\frac{g}{a}} \qquad g = \det(g_{ij}) \qquad a = \det(a_{\alpha\beta}) \tag{7.3}$$

In Eqs (7.1) the quantities $\overset{0}{p}{}^{\alpha}$, $\overset{0}{p}{}^{3}$, are 0-order couples of the surface loads defined below (in the case of thin shells) as follows

$$\overset{0}{p}{}^{\alpha} = {}^{*}t^{3\alpha}\Big|_{-h/2}^{+h/2} \qquad \overset{0}{p}{}^{3} = {}^{*}t^{33}\Big|_{-h/2}^{+h/2} \tag{7.4}$$

where ${}^{*}t^{3\beta}$, ${}^{*}t^{33}$ are the boundary values of components of the first Piola-Kirchhoff stress tensor, balancing the applied to the upper and lower faces of the shell external surface loadings. In the papers [1 ÷ 3] the external loadings of first order do not occur, so we put $\overset{1}{p}{}^{\alpha} \equiv 0$.

In order to compare the quantities (6.4) with their counterparts (7.4) we should take into account that $t^{bi} = \frac{\partial x^i}{\partial \Theta^a} s^{ba}$, where $\left[\frac{\partial x^i}{\partial \Theta^a}\right]$ is a placement of the mapping

$$(\Theta^1, \Theta^2, \Theta^3) \to \Big\{x^1(\Theta^1,\Theta^2,\Theta^3), x^2(\Theta^1,\Theta^2,\Theta^3), x^3(\Theta^1,\Theta^2,\Theta^3)\Big\}$$

Moreover, it is possible to show that

$$\frac{\partial x^i}{\partial \Theta^a} = V^j|_a(\Theta) + c^j_a\Big(\Theta, x(\Theta)\Big) \qquad [c^j_a] \cong \mathsf{I}$$

For thin shells we obtain

$$t^{3\beta} = \frac{\partial x^\beta}{\partial \Theta^a} s^{3a} \cong \frac{\partial x^\beta}{\partial \Theta^\alpha} s^{3\alpha} \Leftrightarrow S^{3\beta}$$

$$t^{33} = \frac{\partial x^3}{\partial \Theta^a} s^{3a} = \frac{\partial x^3}{\partial \Theta^\alpha} s^{3\alpha} + \frac{\partial x^3}{\partial \Theta^3} s^{33} \Leftrightarrow S^{33} + S^{3\Gamma} W|_\Gamma$$

Involving the estimations $(5.5)_1$ and (5.16) one arrives at the equivalence $P^\Gamma \Leftrightarrow \overset{0}{p}{}^\alpha$ and $P \Leftrightarrow \overset{0}{p}{}^3$.

Let us notice that in $[1 \div 3]$ the equilibrium equations (6.2) have been obtained for $\beta_3 = 0$, cf [5]. In the notations of [4] it means that

$$\overset{1}{v}_3 = 0 \tag{7.5}$$

With regard to this dependence the set of substitutions (7.2) is a simplified form of the corresponding original relations given by Schmidt and Weichert [4].

Obviously, all the assumptions $(\mathbf{A}_1) \div (\mathbf{A}_6)$ listed in Section 5 are compulsory here.

Now, we pass to deriving a particular case of the Eqs (7.1) which is considered here as the case **(a)** and has been studied by Duszek $[1 \div 3]$.

First, we evaluate a few quantities appearing in Eqs (7.2). We put $\overset{2}{L}{}^{\alpha\beta} = 0$ and $\overset{1}{L}{}^{\beta 3} = 0$ because in $[1 \div 3]$ these moments (bending 2nd order and torsional 1st order) do not occur. Formulas (4.13) and (5.12) imply that $\overset{1}{\varphi}_\alpha = 0$ and $\overset{0}{\varphi}_\alpha = \overset{0}{v}_{3,\alpha} = O(\varepsilon)$.

Let us recall yet the evaluation $B^{\Sigma}_{\Delta} = b^\delta_\sigma = O(1/R) = O(\varepsilon^2)h$, cf $[1 \div 3]$, [5]. With regard to this dependence, the second component in the expression for $\overset{0}{S}{}^{\alpha\beta}$ may be omitted.

Let us analyse now the quantity $\overset{0}{S}{}^{3\beta}$ appearing in Eqs (7.1). We realize that

$$\overset{0}{S}{}^{3\beta} = \overset{0}{\dot{L}}{}^{\beta 3} + (\overset{0}{\varphi}_\lambda \overset{0}{L}{}^{\lambda\beta})^{\cdot}$$

In such a form it occurs in Eq $(7.1)_2$ but in Eq $(7.1)_1$ only the first component: $\overset{0}{\dot{L}}{}^{\beta 3}$ remains, as in the expression $b^\alpha_\beta \overset{0}{S}{}^{3\beta}$ the second one has the order of magnitude $O(\varepsilon^3)$. Now we estimate the second term, i.e. $b_{\alpha\beta} \overset{0}{S}{}^{\alpha\beta}$, occuring in Eq $(7.1)_2$. We have

$$\begin{aligned} b_{\alpha\beta} \overset{0}{S}{}^{\alpha\beta} &= b_{\alpha\beta} \overset{0}{\dot{L}}{}^{\alpha\beta} + b_{\alpha\beta} g^{\alpha\sigma} \overset{1}{v}_\sigma \overset{0}{\dot{L}}{}^{\beta 3} = \\ &= b_{\alpha\beta} \overset{0}{\dot{L}}{}^{\alpha\beta} + b^\sigma_\beta \overset{1}{v}_\sigma \overset{0}{\dot{L}}{}^{\beta 3} = b_{\alpha\beta} \overset{0}{\dot{L}}{}^{\alpha\beta} + O(\varepsilon^2)O(\varepsilon) \overset{0}{\dot{L}}{}^{\beta 3} \end{aligned}$$

It is shown here that only the first term of the above expression should be preserved in Eq $(7.1)_2$.

Let us pass now to Eq $(7.1)_3$. Obviously we have: $\overset{1}{S}{}^{\alpha\beta} = \overset{1}{\dot{L}}{}^{\alpha\beta}$. The second component in this equation, i.e. $b^\alpha_\beta \hat{S}^{3\beta} = O(\varepsilon^2)O(\varepsilon)$ ought to be neglected.

The last quantity we have to estimate is $\overset{1}{R}{}^{\alpha 3}$. It is easy to notice, see Eqs (4.13), that $\overset{0}{\varphi}{}^{\alpha}_{\lambda} = O(\varepsilon^2)$. So the second and third terms in the expression for $\overset{1}{R}{}^{\alpha 3}$ are negligible. Then we obtain: $\overset{1}{R}{}^{\alpha 3} = -\overset{0}{\dot{L}}{}^{\alpha 3}$.

Now we have already completed the analysis of equilibrium equations. We can compare only first three equations of the set (7.1), with regard to the condition $\overset{1}{v}_3 = 0$, see Eqs (7.4); hence also $\delta\, \overset{1}{v}_3 = 0$.

We put in Eqs (7.1) $\overset{1}{p}{}^{\alpha} \equiv 0$ because in [1 ÷ 3] such an external loading does not occur.

Now, after the above evaluations of many quantities of the formulas (7.2) and Eqs (7.1), we can write down the equilibrium equations derived here from the result obtained by Schmidt and Weichert [4] in the case (a) considered by Duszek [1 ÷ 3], as follows

$$
\begin{aligned}
&\delta\, \overset{0}{\dot{v}}_{\alpha}: \quad \overset{0}{\dot{L}}{}^{\alpha\beta}|_{\beta} + (\overset{1}{v}{}^{\alpha}\, \overset{0}{\dot{L}}{}^{\beta 3})|_{\beta} + (\overset{1}{\dot{v}}{}^{\alpha}\, \overset{0}{L}{}^{\beta 3})|_{\beta} - b^{\alpha}_{\beta}\, \overset{0}{\dot{L}}{}^{\beta 3} + \overset{0}{\dot{p}}{}^{\alpha} = 0 \\
&\delta\, \overset{0}{\dot{v}}_{3}: \quad \overset{0}{\dot{L}}{}^{3\beta}|_{\beta} + (\overset{0}{v}_{3,\lambda}\, \overset{0}{\dot{L}}{}^{\lambda\beta})|_{\beta} + (\overset{0}{\dot{v}}_{3,\lambda}\, \overset{0}{L}{}^{\lambda\beta})|_{\beta} + b_{\alpha\beta}\, \overset{0}{\dot{L}}{}^{\alpha\beta} + \overset{0}{\dot{p}}{}^{3} = 0 \qquad (7.6) \\
&\delta\, \overset{1}{\dot{v}}_{\alpha}: \quad \overset{1}{\dot{L}}{}^{\alpha\beta}|_{\beta} - \overset{0}{\dot{L}}{}^{\alpha 3} = 0
\end{aligned}
$$

where $\overset{1}{v}_{\alpha} = -\overset{0}{v}_{3,\alpha}$.

Nowadays, it is clear that the equilibrium equations (7.6) derived here from the results which were got by Schmidt and Weichert at moderate rotations of shells are the same as the equations obtained by Duszek at moderate deflections. It was possible to prove this equivalency for shallow shells, because for such a kind of shells these equations were given by Duszek [1 ÷ 3] explicitly.

It is possible to verify that static boundary conditions, obtained by Duszek [1÷3] from the principle of virtual work, are identical with those ones presented by Schmidt and Weichert [4].

8. Concluding remarks

Two approaches to geometrically nonlinear plastic shell problem have been compared. The kinematical hypotheses assumed there are weaker than the assumptions of the Kirchhoff-Love theory.

Under appropriate simplifications imposed on the kinematical and statical quantities, involved in the model studied by Schmidt and Weichert [4], the three classes of geometrical relations for shells at moderate deflections, obtained by Duszek [1 ÷ 3], are derived here from the theory of shells at moderate rotations, [4]. In what concerns equivalent equations the coincidence has been demonstrated for shallow shells only.

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Porównanie dwu teorii geometrycznie nieliniowych powłok

Streszczenie

W pracy dokonano porównania dwu sformnułowań problemu geometrycznie nieliniowych powlok. Rozpatrzono i porównano tensory odkształcenia powloki oraz równania równowagi otrzymane przez Duszek [1÷3] oraz Schmidta i Weicherta [4]. Wykazano, że trzy spośród wielu teorii uzyskanych przez Duszek [1÷3] są szczególnymi przypadkami teorii zaproponowanej przez Schmidta i Weicherta [4].

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