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THE NON-LINEAR VIBRATIONS OF PARAMETRICALLY SELF-EXCITED SYSTEM WITH TWO DEGREES OF FREEDOM

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Non-linear vibrations of a system with two degrees of freedom with parametric and self-excited vibrations interaction were investigated in this paper. The analysis was carried out for the main parametric resonance in the neighbourhood of the first and second frequencies of free vibrations. The free vibrations amplitudes and the widths of the synchronization areas were calculated analytically. Stability of the obtained periodic solutions was examined as well. The analytical results were verified and supplemented with the analog and computer simulation.

1. Introduction

The class of system vibrations of which are described by differential equations with parametric and self-excitation terms exists in machine dynamics. A majority of papers analysing the interaction between self-excited and parametric vibrations concern systems with one degree of freedom. Minorsky (1967) was one the first who tried to carry out the analysis of self-excited oscillator vibrations with periodically changing parameters.

Kononenko and Kovalchuk (1971a) examined the influence of parametric excitation and the friction model applied on the amplitude of one degree of freedom system vibrations in the main parametric resonance area. Kononenko and Kovalchuk (1971b) considered parametrically self-excited system vibrations under parallel parametric external excitation. Bolotin (1984) examined the resonance states in parametrically self-excited systems represented by equations containing Van der Pol-Duffing-Mathieu and Rayleigh-Duffing-Mathieu terms, respectively. Tondl (1978) analysed parametrically self-excited system vibrations. He determined the synchronization areas and vibration amplitudes in these areas. He also suggested method enabling one to investigate vibrations outside the synchronization area. The analysis was made upon systems vibrating under "soft" and "hard" self-excitations, respectively. The analog simulation verified analytically obtained results with one or two degrees of freedom. Yano et al. (1986) examined the parametrically self-excited system investigating the oscillator with one degree of freedom (Van der Pol-Mathieu type). Yano (1984), (1987) and (1989) analysed interactions between self-excited and parametric vibrations taking various types of non-linearity into account. The amplitude curves were plotted; some system parameters influence on the vibrations amplitude and synchronization area width was detected. The stability solutions was examined applying the Ruth-Hurwitz criterion. Szabelski (1984) and (1991) examined self-excited vibrating systems represented by equations with Van der Pol's term, under parametric Mathieu-type excitation with non-linear symmetric and non-symmetric characteristics of system elasticity. The stroboscope phase surface method and analog simulation were applied. Alifov and Frolov (1985) examined the interaction between parametrically self-excited systems with ideal energy sources. The parametrically self-excited continuous system vibrations were shown by Dżygadlo (1972). Transverse vibrations of rotating shaft pinned at both ends in the main resonance area were investigated.

The review presented above proves that investigation into the self-excited-parametric system vibrations concerned mainly one-degree-of-freedom systems. The present paper deals with two-degree-of-freedom systems. Since the Van der Pol model (cf Kauderer (1958)) has no use in mechanics, the Rayleigh model of self-excited vibration generating has been applied.

2. Mathematical model; differential equations of motion in quasi-normal coordinates

Let us consider a parametrically self-excited system with two degrees of freedom manifesting non-linear, symmetric elasticity characteristics. Let us assume that the Mathieu-type parametric excitation appears and non-linear damping is represented by the Rayleigh model. The differential equations of motion in generalized coordinates are as follows

$$
\begin{aligned}
&m_1\ddot{z}_1 + (c_0 - c_1\cos 2\nu t)(z_1 - z_2) + k_0 z_1 + k_1 z_1^3 = 0 \\
&m_2\ddot{z}_2 + (a - b\dot{z}_2^2)\dot{z}_2 - (c_0 - c_1\cos 2\nu t)(z_1 - z_2) = 0
\end{aligned}
\tag{2.1}
$$

where

z_1, z_2	–	generalized coordinates
k_0, c_0, k_0	–	rigidity coefficients
k_1	–	coefficient describing the non-linear part of elastic force.

A sample physical model of the system with two degrees of freedom is shown in Fig.1.

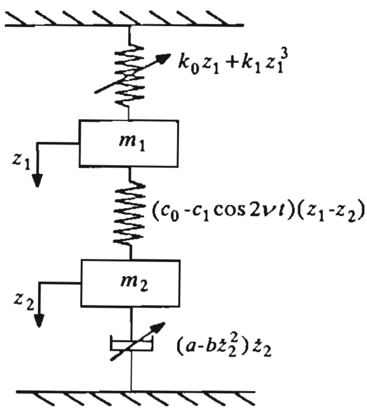

Fig. 1. Physical model of a parametrically-self-excited system with two degrees of freedom

A non-dimensional term τ can be introduced to Eqs (2.1) substituting

$$\tau = \omega_1 t$$

where $\omega_1 = \sqrt{k_0/m_1}$.

Introducing into Eqs (2.1) the following notation

$$
\begin{array}{llll}
v = \dfrac{\nu}{\omega_1} & \lambda = \dfrac{c_0}{k_0} & M = \dfrac{m_1}{m_2} & z_0 = \dfrac{m_1 g}{k_0} \\[2ex]
\alpha = \dfrac{a}{m_2\omega_1} & \beta = \dfrac{b\omega_1}{m_2} z_0^2 & \gamma = \dfrac{k_1}{k_0} z_0^2 & \mu = \dfrac{c_1}{c_0} \\[2ex]
x_1 = \dfrac{z_1}{z_0} & x_2 = \dfrac{z_2}{z_0} & \dot{x}_1 = \dfrac{dx_1}{d\tau} & \dot{x}_2 = \dfrac{dx_2}{d\tau}
\end{array}
$$

$$\ddot{x}_1 = \frac{d^2 x_1}{d\tau^2} \qquad \ddot{x}_2 = \frac{d^2 x_2}{d\tau^2}$$

we obtain the non-dimensional differential equations system

$$\begin{aligned} &\ddot{x}_1 + \lambda(1 - \mu\cos 2\upsilon\tau)(x_1 - x_2) + x_1 + \gamma x_1^3 = 0 \\ &\ddot{x}_2 - (\alpha - \beta\dot{x}_2^2)\dot{x}_2 - \lambda M(1 - \mu\cos 2\upsilon\tau)(x_1 - x_2) = 0 \end{aligned} \tag{2.2}$$

Assuming that the coefficients α, β, γ, μ in Eqs (2.2) are small and positive and $\alpha = \mu\bar{\alpha}$, $\beta = \mu\bar{\beta}$, $\gamma = \mu\bar{\gamma}$ we have

$$\begin{aligned} &\ddot{x}_1 + \lambda(x_1 - x_2) + x_1 = \mu[\lambda(x_1 - x_2)\cos 2\upsilon\tau - \bar{\gamma}x_1^3] \\ &\ddot{x}_2 - \lambda M(x_1 - x_2) = \mu[(\bar{\alpha} - \bar{\beta}\dot{x}_2^2) - \lambda M(x_1 - x_2)\cos 2\upsilon\tau] \end{aligned} \tag{2.3}$$

For $\mu = 0$ Eqs (2.3) represent free vibrations of a linear system with two degrees of freedom. We transform Eqs (2.3) to quasi-normal coordinates y_1, y_2, for witch at $\mu = 0$ the system of equation is uncoupled into two independent differential equations

$$\begin{aligned} \ddot{y}_1 + p_1^2 y_1 &= \mu\Big\{-\delta_1 p_1^2 \cos 2\upsilon\tau(\varepsilon_1 y_2 - \varepsilon_2 y_1) - \bar{\gamma}(\psi_1 y_2 - \psi_2 y_1)^3 + \\ &+ \delta_1\varphi(\dot{y}_1 - \dot{y}_2)\Big[\bar{\alpha} - \bar{\beta}\varphi^2(\dot{y}_1 - \dot{y}_2)^2\Big]\Big\} \\ \ddot{y}_2 + p_2^2 y_2 &= \mu\Big\{-\delta_2 p_1^2 \cos 2\upsilon\tau(\varepsilon_1 y_2 - \varepsilon_2 y_1) - \bar{\gamma}(\psi_1 y_2 - \psi_2 y_1)^3 + \\ &+ \delta_2\varphi(\dot{y}_1 - \dot{y}_2)\Big[\bar{\alpha} - \bar{\beta}\varphi^2(\dot{y}_1 - \dot{y}_2)^2\Big]\Big\} \end{aligned} \tag{2.4}$$

where free vibration frequencies of the linear system are

$$p_{1,2}^2 = \frac{1}{2}\Big[1 + \lambda + M\lambda \mp \sqrt{(1 + \lambda + M\lambda)^2 - 4M\lambda}\Big]$$

and

$$\delta_1 = \frac{1 + \lambda - p_1^2}{\lambda M} \qquad \delta_2 = \frac{1 + \lambda - p_2^2}{\lambda M}$$

$$\psi_1 = \frac{\delta_1}{\delta_1 - \delta_2} \qquad \psi_2 = \frac{\delta_2}{\delta_1 - \delta_2} \qquad \varphi = \frac{1}{\delta_1 - \delta_2}$$

$$\varepsilon_1 = \psi_1 + \varphi \qquad \varepsilon_2 = \psi_2 + \varphi$$

Let us assume that the free vibration frequencies ratio p_2/p_1 value is not a natural number, what means that internal resonance does not appear. Relations between generalized and quasi-normal coordinates are as follows

$$y_1 = x_1 + \delta_1 x_2 \qquad\qquad y_2 = x_1 + \delta_2 x_2 \tag{2.5}$$

3. Periodic vibrations in the main parametric resonance area with respect to the first and second free vibrations frequencies

In the first approximation we assume the solution to Eqs (2.4) in the first free vibration frequency neighbourhood in the following form

$$\begin{aligned} y_1 &= B_1(\tau)\cos v\tau + B_2(\tau)\sin v\tau \\ y_2 &= 0 \end{aligned} \tag{3.1}$$

where $B_1(\tau)$ and $B_2(\tau)$ – slowly changing functions of time.

Substituting Eqs (3.1) into Eqs (2.4) and assuming

$$\Big(\frac{dB_1(\tau)}{d\tau}\Big)^3 \approx 0 \qquad\qquad \Big(\frac{dB_1(\tau)}{d\tau}\Big)^2 \approx 0$$

$$\frac{dB_1(\tau)}{d\tau}\frac{dB_2(\tau)}{d\tau} \approx 0$$

we obtain the system of approximate differential equations of the first order

$$\begin{aligned} &\frac{dB_1}{d\tau}\varphi\delta_1\Big[-\alpha + \frac{3}{4}\beta\varphi^2 v^2(3B_2^2 + B_1^2)\Big] + \frac{dB_2}{d\tau}v\Big(2 - \frac{3}{2}\delta_1\beta\varphi^3 v B_1 B_2\Big) + \\ &+B_1\Big[-(v^2 - p_1^2) - \frac{1}{2}\mu\delta_1 p_1^2\varepsilon_2 - \frac{3}{4}\gamma\psi_2^3 R_1^2\Big] + B_2\varphi v\delta_1\Big(-\alpha + \frac{3}{4}\beta\varphi^2 v^2 R_1^2\Big) = 0 \\ &\frac{dB_1}{d\tau}v\Big(2 + \frac{3}{2}\delta_1\beta\varphi^3 v B_1 B_2\Big) + \frac{dB_2}{d\tau}\varphi\delta_1\Big[-\alpha - \frac{3}{4}\beta\varphi^2 v^2(B_2^2 + 3B_1^2)\Big] + \\ &+B_1\varphi v\delta_1\Big(-\alpha + \frac{3}{4}\beta\varphi^2 v^2 R_1^2\Big) + B_2\Big[(v^2 - p_1^2) - \frac{1}{2}\mu\delta_1 p_1^2\varepsilon_2 + \frac{3}{4}\gamma\psi_2^3 R_1^2\Big] = 0 \end{aligned} \tag{3.2}$$

where $R_1^2 = B_1^2 + B_2^2$.

For the steady state $dB_1(\tau)/d\tau = 0$ and $dB_2(\tau)/d\tau = 0$ from Eqs (3.2) we obtain the following system of algebraic, non-linear equations

$$
\begin{aligned}
&B_1\Big[-(v^2-p_1^2)-\frac{1}{2}\mu\delta_1 p_1^2\varepsilon_2-\frac{3}{4}\gamma\psi_2^3R_1^2\Big]+B_2\varphi v\delta_1\Big(-\alpha+\frac{3}{4}\beta\varphi^2v^2R_1^2\Big)=0\\
&B_1\varphi v\delta_1\Big(-\alpha+\frac{3}{4}\beta\varphi^2v^2R_1^2\Big)+B_2\Big[(v^2-p_1^2)-\frac{1}{2}\mu\delta_1p_1^2\varepsilon_2+\frac{3}{4}\gamma\psi_2^3R_1^2\Big]=0
\end{aligned}
\tag{3.3}
$$

From Eqs (3.3) we can write the equation representing the resonance curve

$$
\begin{aligned}
&\frac{9}{16}R_1^4(\gamma^2\psi_2^6+\delta_1^2\varphi^6v^6\beta^2)+\frac{3}{2}R_1^2\Big[\gamma\psi_2^3(v^2-p_1^2)-\delta_1^2\varphi^4v^4\alpha\beta\Big]+\\
&+(v^2-p_1^2)^2+\delta_1^2\Big(\alpha^2\varphi^2v^2-\frac{1}{4}\mu^2\varepsilon_2^2p_1^4\Big)=0
\end{aligned}
\tag{3.4}
$$

Then we determine vibrations amplitude for the main parametric resonance area lying in the first free vibration frequency neighbourhood

$$
R_1^2=\frac{-4\Big[\gamma\psi_2^3(v^2-p_1^2)-\delta_1^2\varphi^4v^4\alpha\beta\Big]\mp\sqrt{\Delta}}{3(\gamma^2\psi_2^6+\delta_1^2\varphi^6v^6\beta^2)}
\tag{3.5}
$$

Putting $R_1 = 0$ into Eq (3.4) we find the frequencies corresponding to the bifurcation points, at which the amplitude-frequency curves separate from the trivial solution

$$
v_{1,2}=\frac{1}{2}\Big[2p_1^2-\delta_1^2\alpha^2\varphi^2\mp\sqrt{(\delta_1^2\alpha^2\varphi^2-2p_1^2)^2-4p_1^4\Big(1-\frac{1}{4}\delta_1^2\mu^2\varepsilon_2^2\Big)}\Big]
\tag{3.6}
$$

Upon assuming that $\delta_1^4\alpha^4\varphi^4\approx 0$ the condition of trivial solutions bifurcation into non-trivial ones takes the form

$$
\frac{\mu}{\alpha}>2\frac{\varphi}{p_1\varepsilon_2}
\tag{3.7}
$$

For the resonance with respect to the second free vibration frequency the solution to Eqs (2.4) takes the form

$$
\begin{aligned}
y_1&=0\\
y_2&=B_3(\tau)\cos v\tau+B_4(\tau)\sin v\tau
\end{aligned}
\tag{3.8}
$$

where: $B_3(\tau)$ and $B_4(\tau)$ – slowly changing functions of time.

Similarly as in the main parametric resonance area relatively to frequency p_1, we get the system of approximate differential equations of the first order

$$\begin{aligned}
&\frac{dB_3}{d\tau}\varphi\delta_2\Big[\alpha-\frac{3}{4}\beta\varphi^2v^2(3B_4^2+B_3^2)\Big]+\frac{dB_4}{d\tau}v\Big(2+\frac{3}{2}\delta_2\beta\varphi^3vB_3B_4\Big)+\\
&+B_3\Big[-(v^2-p_2^2)+\frac{1}{2}\mu\delta_2p_2^2\varepsilon_1+\frac{3}{4}\gamma\psi_1^3R_2^2\Big]+B_4\varphi v\delta_2\Big(\alpha-\frac{3}{4}\beta\varphi^2v^2R_2^2\Big)=0\\
&\frac{dB_3}{d\tau}v\Big(2-\frac{3}{2}\delta_2\beta\varphi^3vB_3B_4\Big)+\frac{dB_4}{d\tau}\varphi\delta_2\Big[-\alpha+\frac{3}{4}\beta\varphi^2v^2(B_4^2+3B_3^2)\Big]+\\
&+B_3\varphi v\delta_2\Big(\alpha-\frac{3}{4}\beta\varphi^2v^2R_2^2\Big)+B_4\Big[(v^2-p_2^2)+\frac{1}{2}\mu\delta_2p_2^2\varepsilon_1-\frac{3}{4}\gamma\psi_1^3R_2^2\Big]=0
\end{aligned} \tag{3.9}$$

where $R_2^2=B_3^2+B_4^2$.

For the stable state we have

$$\begin{aligned}
&B_3\Big[-(v^2-p_2^2)+\frac{1}{2}\mu\delta_2p_2^2\varepsilon_1+\frac{3}{4}\gamma\psi_1^3R_2^2\Big]+B_4\varphi v\delta_2\Big(\alpha-\frac{3}{4}\beta\varphi^2v^2R_2^2\Big)=0\\
&B_3\varphi v\delta_2\Big(\alpha-\frac{3}{4}\beta\varphi^2v^2R_2^2\Big)+B_4\Big[(v^2-p_2^2)+\frac{1}{2}\mu\delta_2p_2^2\varepsilon_1-\frac{3}{4}\gamma\psi_1^3R_2^2\Big]=0
\end{aligned} \tag{3.10}$$

The resonance curve equation has the form

$$\begin{aligned}
&\frac{9}{16}R_2^4(\gamma^2\psi_1^6+\delta_2^2\varphi^6v^6\beta^2)+\frac{3}{2}R_2^2\Big[-\gamma\psi_1^3(v^2-p_2^2)-\delta_2^2\varphi^4v^4\alpha\beta\Big]+\\
&+(v^2-p_2^2)^2+\delta_2^2\Big(\alpha^2\varphi^2v^2-\frac{1}{4}\mu^2\varepsilon_1^2p_2^4\Big)=0
\end{aligned} \tag{3.11}$$

and the vibrations amplitude is a follows

$$R_2^2=\frac{-4\Big[\gamma\psi_1^3(v^2-p_2^2)-\delta_2^2\varphi^4v^4\alpha\beta\Big]\mp\sqrt{\Delta}}{3(\gamma^2\psi_1^6+\delta_2^2\varphi^6v^6\beta^2)} \tag{3.12}$$

where

$$\begin{aligned}
\Delta &= \Big[-\gamma\psi_1^3(v^2-p_2^2)-\delta_2^2\varphi^4v^4\alpha\beta\Big]^2+\\
&- (\gamma^2\psi_1^6+\delta_2^2\varphi^6v^6\beta^2)\Big[-(v^2-p_2^2)^2-\delta_2^2\Big(\alpha^2\varphi^2v^2-\frac{1}{4}\mu^2\varepsilon_1^2p_2^4\Big)\Big]
\end{aligned}$$

Substituting for $R_2=0$ into Eq (3.11) we find the bifurcation points at which the trivial solutions are changed into non-trivial ones

$$v_{1,2}=\frac{1}{2}\Big[2p_2^2-\delta_2^2\alpha^2\varphi^2\mp\sqrt{(\delta_2^2\alpha^2\varphi^2-2p_2^2)^2-4p_2^4\Big(1-\frac{1}{4}\delta_2^2\mu^2\varepsilon_1^2\Big)}\Big] \tag{3.13}$$

and the bifurcation existence condition takes the form

$$\frac{\mu}{\alpha} > 2\frac{\varphi}{p_2\varepsilon_1} \tag{3.14}$$

4. Periodic vibration stability

The periodic vibration stability examination was carried out applying Eqs (3.2) to the case of resonance with respect to the first free vibration frequency and Eqs (3.9) to the case of resonance relative to the second free vibration frequency. Eqs (3.2) and (3.9) can be rewritten as

$$\begin{aligned} \frac{dB_i}{d\tau}f_{11} + \frac{dB_j}{d\tau}f_{12} &= f_{13} \\ \frac{dB_i}{d\tau}f_{21} + \frac{dB_j}{d\tau}f_{22} &= f_{23} \end{aligned} \tag{4.1}$$

where $i = 1$, $j = 2$ for Eqs (3.2) and $i = 3$, $j = 4$ for Eqs (3.9).

Transforming Eqs (4.1) to the form

$$\begin{aligned} \frac{dB_i}{d\tau} &= \frac{f_{13}f_{22} - f_{12}f_{23}}{f_{11}f_{22} - f_{12}f_{21}} = F_1(B_i, B_j) \\ \frac{dB_j}{d\tau} &= \frac{f_{11}f_{23} - f_{13}f_{21}}{f_{11}f_{22} - f_{12}f_{21}} = F_2(B_i, B_j) \end{aligned} \tag{4.2}$$

and introducing disturbances into Eqs (4.2)

$$\begin{aligned} \widetilde{B}_i(\tau) &= B_i(\tau) + \delta_{B_i} \\ \widetilde{B}_j(\tau) &= B_j(\tau) + \delta_{B_j} \end{aligned} \tag{4.3}$$

where $\widetilde{B}_i(\tau)$ and $\widetilde{B}_j(\tau)$ – solutions corresponding to the disturbed initial conditions. Substituting Eqs (4.3) into Eqs (3.2) and (3.9), non-disturbed equations from the disturbed ones and then neglecting small terms of higher order, we find linear differential equations in variations

$$\begin{aligned} \frac{d\delta_{B_i}}{d\tau} &= \Big(\frac{\partial F_1}{\partial B_i}\Big)_0 \delta_{B_i} + \Big(\frac{\partial F_1}{\partial B_j}\Big)_0 \delta_{B_j} \\ \frac{d\delta_{B_j}}{d\tau} &= \Big(\frac{\partial F_2}{\partial B_i}\Big)_0 \delta_{B_i} + \Big(\frac{\partial F_2}{\partial B_j}\Big)_0 \delta_{B_j} \end{aligned} \tag{4.4}$$

where: $\left(\frac{\partial F_1}{\partial B_i}\right)_0$, $\left(\frac{\partial F_1}{\partial B_j}\right)_0$, $\left(\frac{\partial F_2}{\partial B_i}\right)_0$, $\left(\frac{\partial F_2}{\partial B_j}\right)_0$ – partial derivatives at the equilibrium point (index 0). The characteristic equation (4.4) has the following roots

$$\zeta_{1,2} = \frac{1}{2}\Big(H_1 \mp \sqrt{H_1^2 - 4H_2}\Big) \tag{4.5}$$

where

$$H_1 = \left(\frac{\partial F_1}{\partial B_i}\right)_0 + \left(\frac{\partial F_2}{\partial B_j}\right)_0$$

$$H_2 = \left(\frac{\partial F_1}{\partial B_i}\right)_0\left(\frac{\partial F_2}{\partial B_j}\right)_0 - \left(\frac{\partial F_1}{\partial B_j}\right)_0\left(\frac{\partial F_2}{\partial B_i}\right)_0$$

$i = 1, j = 2$ for the resonance relatively to the frequency p_1 and $i = 3, j = 4$ for the resonance relatively to the frequency p_2.

The stability approximate solutions depends on the roots of Eqs (4.4).

5. Analytical investigations; analog and computer simulation results

Some sample calculations for the system with two degrees of freedom vibrations were made with the following data (cf Tondl (1978); Szabelski (1984); Yano (1987))

$$\alpha = 0.01 \qquad \beta = 0.05 \qquad \gamma = 0.1$$
$$\mu = 0.2 \qquad M = 0.5 \qquad \lambda = 4$$

Transformation from the generalized to quasi-normal coordinates demands, having in mind the relationships given in Section 2, evaluation of free vibrations frequencies of a linear system p_1, p_2 and the coefficients δ_1, δ_2, ψ_1, ψ_2, ε_1, ε_2, φ.

These values are

$$p_1 = 0.546 \qquad p_2 = 2.589 \qquad \delta_1 = 2.351$$
$$\delta_2 = -0.851 \qquad \psi_1 = 0.734 \qquad \psi_2 = -0.266$$
$$\varepsilon_1 = 1.047 \qquad \varepsilon_2 = 0.047 \qquad \varphi = 0.312$$

The computer and analog simulations were performed using differential equations of motion in generalized coordinates Eqs (2.2). Then, on the ground of Eqs (2.5), transformation into quasi-normal coordinates was caried out.

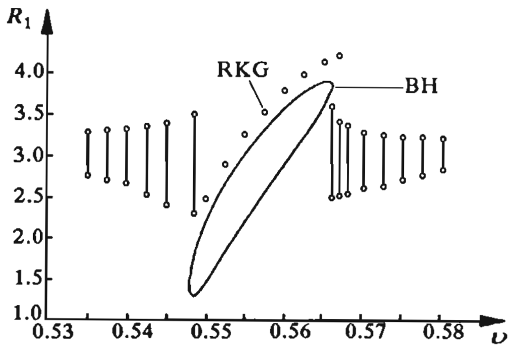

Fig. 2. Vibration amplitudes with respect to the frequency p_1, main parametric resonance, RKG method, BH-analytical investigation

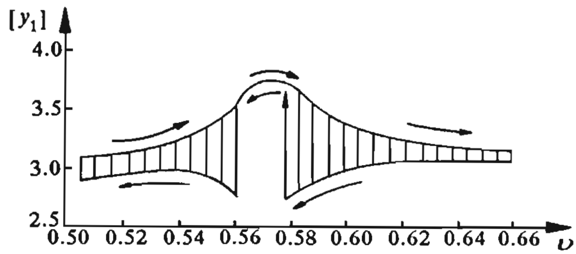

Fig. 3. Vibration amplitudes with respect to the frequency p_1, analog simulation

Eqs (2.2) was solved numerically using the Runge-Kutta-Gill method of fourth order (RKG). The analog simulation was on the MEDA 4500 analog machine.

For the accepted data the amplitude charts were plotted. Fig.2 presents the resonance curve for the free vibrations first frequency. Singular points correspond to stable vibrations amplitudes calculated by means of the RKG method, while the points linked by solid lines represent the minimal and maximal vibration amplitudes respectively in the case of almost periodic vibrations outside the resonance area. A solid line on the chart represents the analytical solution as well. Since the condition (3.7) is not satisfied the resonance curve does not intersect the v axis. Fig.3 presents the analog simulation results for the vibration amplitude registered under slow changes of parametric excitation frequency v. y_1 stands for the amplitudes envelope in quasi-normal coordinates. Arrows point out the direction of parameter v changes. Fig.2 and Fig.3 show that vibrations with stable amplitude occur in a narrow frequency range from $v = 0.55$ to $v = 0.57$ in the main parametric resonance area lying in the p_1 frequency neighbourhood. In this frequency range the

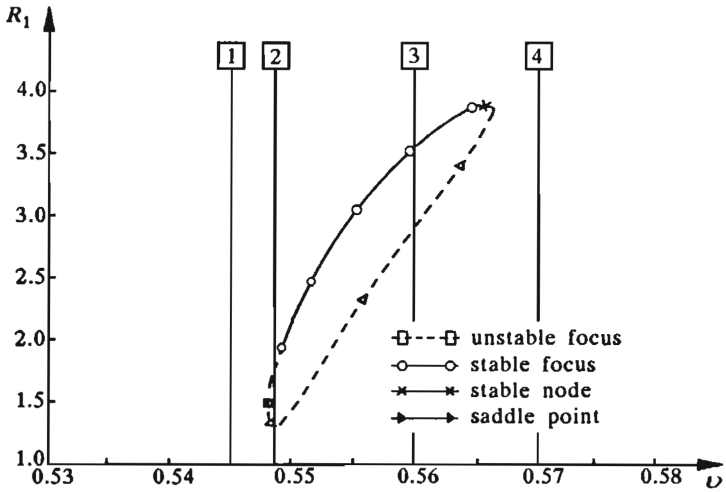

Fig. 4. Vibration amplitudes for the main parametric resonance with respect to the frequency p_1 with the stability type marked out

amplitude is represented as one normal coordinate y_1 only. y_2 coordinate is very small. The vibration amplitudes and synchronization area width determined analytically agree with analog and computer simulation results. The amplitude error determined analytically in the first approximation, comparing to numerical solutions amounts about 5%.

The resonance curve obtained analytically with the stability type marked out is presented in Fig.4. According to the characteristic equation roots, Eq (4.5), different properties of equilibrium states were obtained. In Fig.4, four characteristic values of parameter v are marked: $v_1 = 0.545$, $v_2 = 0.5485$, $v_3 = 0.56$, $v_4 = 0.57$. For these v values, trajectories on the Hayashi's surface $B_1 - B_2$ (Fig.5), time courses in generalized coordinates x_2, x_1 and quasi-normal y_1, y_2 coordinates (Fig.6 and Fig.7), respectively, were plotted. For values v_1 and v_4 the system is outside the main parametric resonance area. For these cases we get stable boundary cycles, which comply with dominant self-excited vibrations, Fig.5a,d for value v_2 two unstable equilibrium states appear unstable focus and saddle, which appear symmetrically in pairs in Fig.5b and a boundary stable cycle. Value v_3 represents the parametric vibrations with two kinds of stability: stable focus and unstable saddle. This is the area where the selfexcited vibrations are pulled in by dominant parametric vibrations, boundary cycle disappears in the Fig.5c.

The influence of the parameter μ on the vibration amplitudes and resonance area width was examined in the example. The dependence $\mu(R_1, v)$

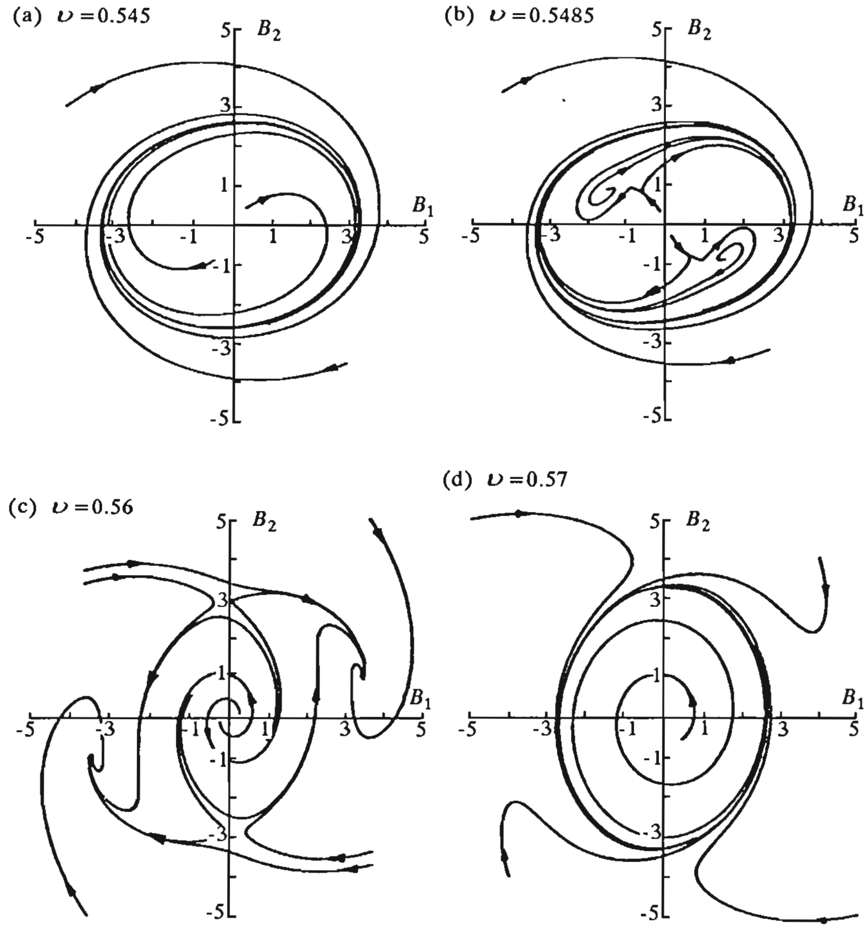

Fig. 5. Integral curves on Hayashi's surface; main parametric resonance with respect to frequency p_1

with the contour lines for $\mu = 0.1$, $\mu = 0.2$, $\mu = 0.3$, $\mu = 0.4$ marked out was presented in Fig.8. Decrease in the value of parameter μ results in reduction of vibrations amplitudes in the main parametric resonance area relative to the frequency p_1 and contraction in this area.

In the case of resonance with respect to the second frequency similar calculations were made. Fig.9 presents vibration amplitudes calculated using the RKG method. Both the vibration amplitudes and the synchronization area are significantly bigger than in the case of resonance with respect to the frequency p_1. Differences between analytical calculations and numerical simulation

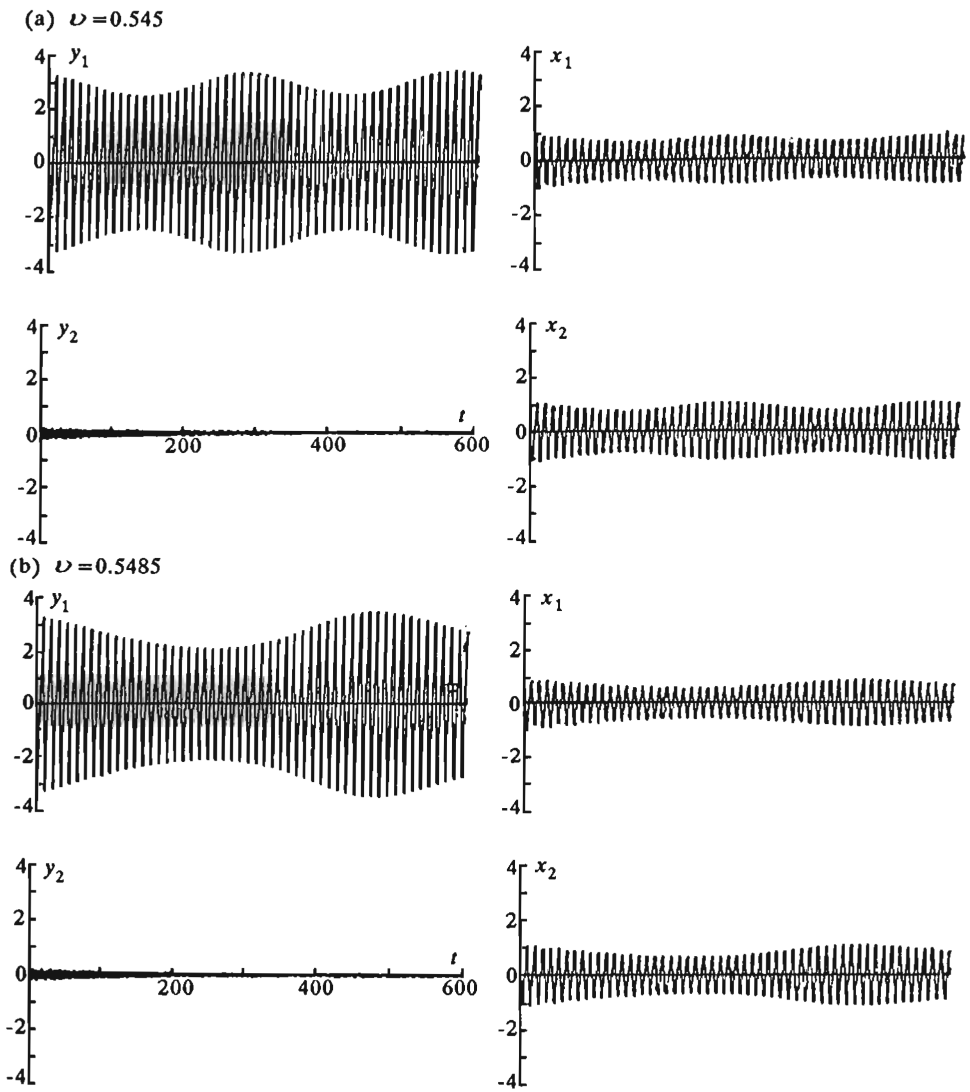

Fig. 6. Vibration time courses in generalized and quasi-normal coordinates; resonance with respect to the frequency p_1

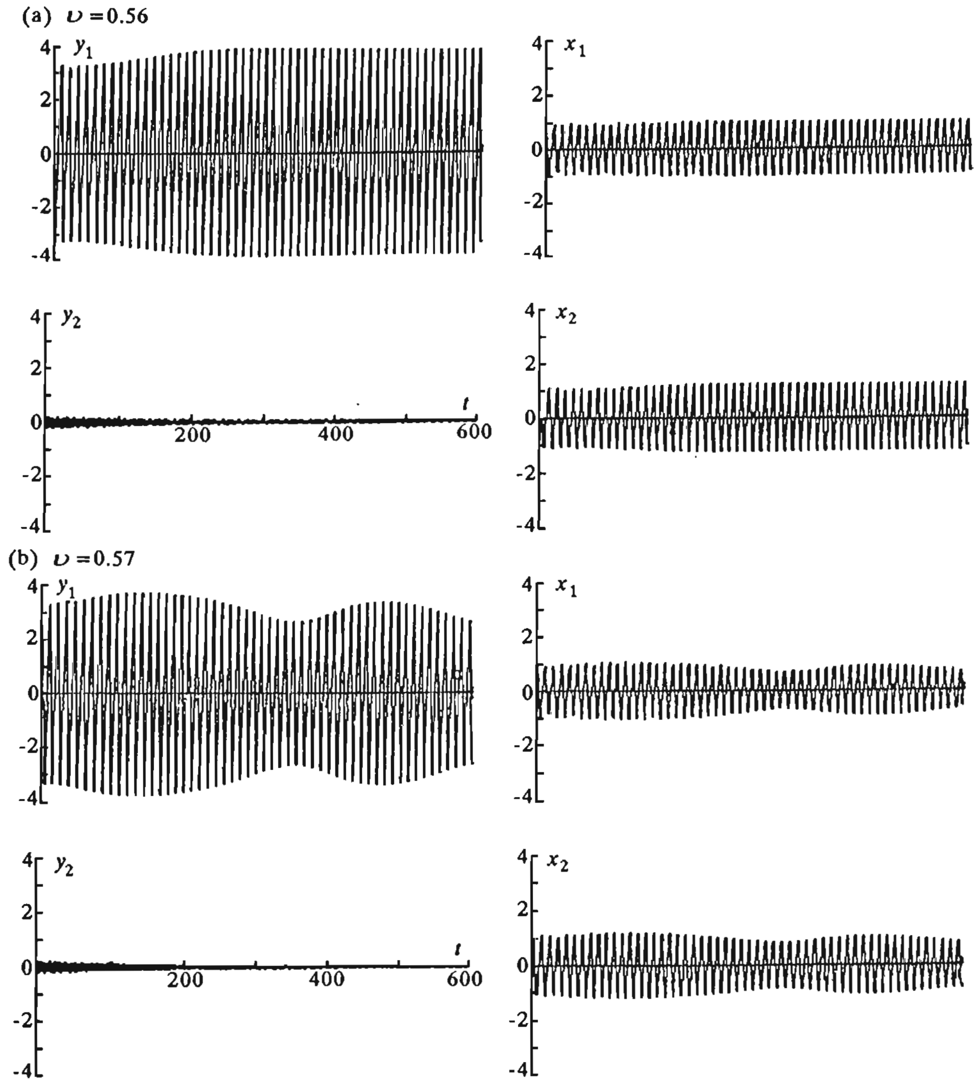

Fig. 7. Vibration time courses in generalized and quasi-normal coordinates; resonance with respect to the frequency p_1

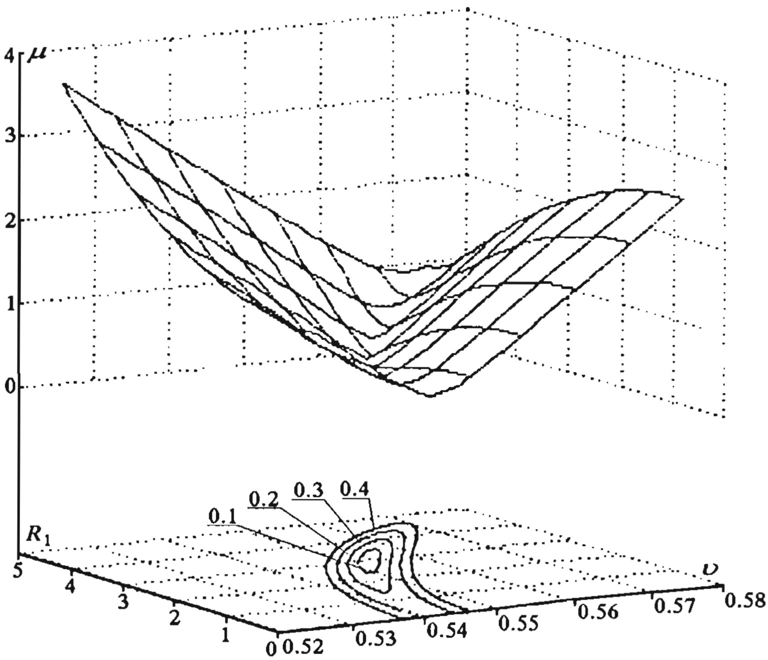

Fig. 8. $\mu(R_1, \upsilon)$ chart for the resonance with respect to the frequency p_1 with the contour lines for $\mu = 0.1$, 0.2, 0.3, 0.4

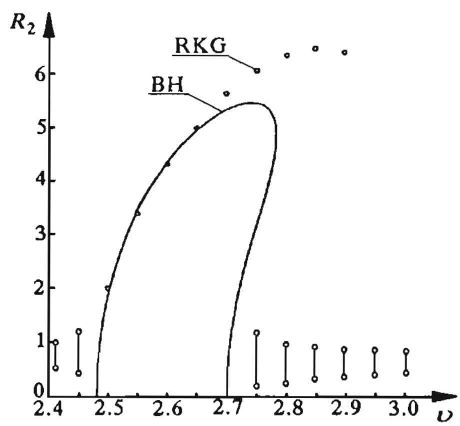

Fig. 9. Vibration amplitudes with respect to the frequency p_2; the main parametric resonance; BH-analytical research

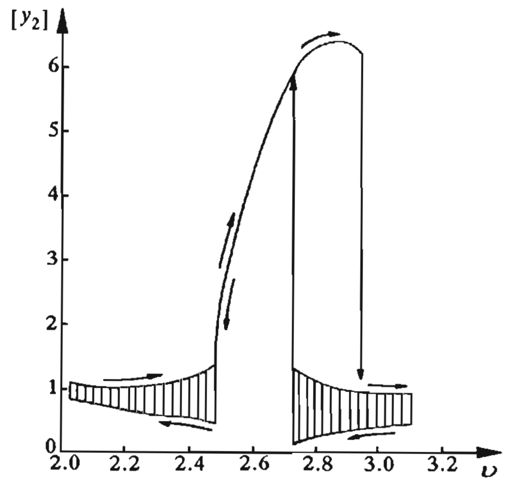

Fig. 10. Vibration amplitudes with respect to the frequency p_2; analog simulation

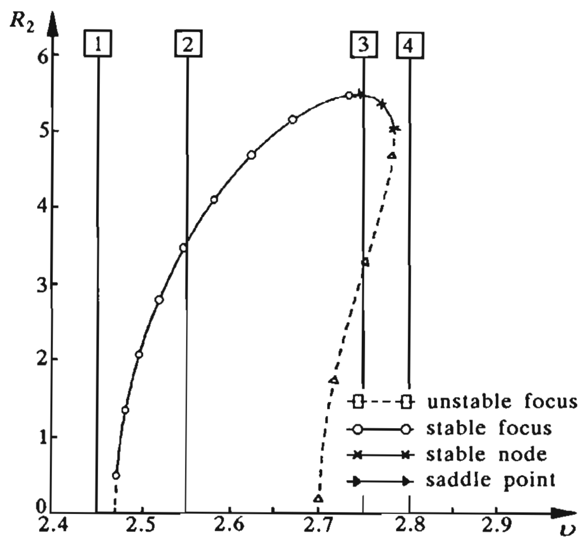

Fig. 11. Vibration amplitudes for the main parametric resonance with respect to the frequency p_2 with marked stability type

results are greater than in the case of resonance with respect to the frequency p_1 (about 15%). Both methods give similar results but only in a narrow area, close to the frequency $p_2 = 2.589$. Results discrepancy grows as a distance from the frequency p_2 increases. Analytical method decreases the synchronization area range width to $v = 2.78$. These differences have a quantitative, not qualitative character.

The analog simulation results are presented in the Fig.10. Increasing v (from $v \approx 2$) we observe almost periodical vibrations with the modulated amplitude. The amplitude modulation grows up to $v \approx 2.48$, when the vibrations synchronization starts; i.e., self-excited vibrations are pulled in by the parametric vibrations. For $v \approx 2.9$ the amplitude break appears and enters the almost periodic vibrations area. The vibration amplitude reaches the value over 6, synchronization area is contained in v values from 2.48 to 2.9.

The analytical resonance curve with the stability kind marked out also four characteristic values of parameter v are shown in Fig.11. For these values v Hayashi's surface $B_3 - B_4$ (Fig.12) and time courses (Fig.13 and Fig.14) are presented, as well.

For the resonance relative to the frequency p_2, the condition (3.14) is fulfilled, that is why the resonance curve intersects the v axis. The bifurcation points at which non-trivial solution became the trivial ones appear for $v_1 = 2.481$, $v_2 = 2.712$. For values v_1 and v_4 we get the stable boundary cycles adequate to the self-excited vibrations occuring outside the synchronization area (Fig.12a,d). Values v_2 and v_3 are inside the synchronization area. For values v_2 we get one equilibrium state in the stable focus (Fig.12b). For values v_3 on the $B_3 - B_4$ surface we get symmetrically paired stable knot, unstable saddle and stable boundary cycle (Fig.12c).

The $\mu(R_2, v)$ surface is shown in Fig.15. The contour lines analysis proves that, similar to the p_1 frequency resonance, with μ parameter increment the synchronization area width and vibration amplitudes grow up. For $\mu^* = 0.00231$ the resonance curve has intersects the v axis only once, while for $\mu < \mu^*$ intersects point do not exist.

6. *Conclusions*

For the system with two degrees of freedom the main parametric resonance appears in quasi-normal coordinates in similar way as for the system with one degree of freedom. In the neighbourhood of p_1 frequency in the

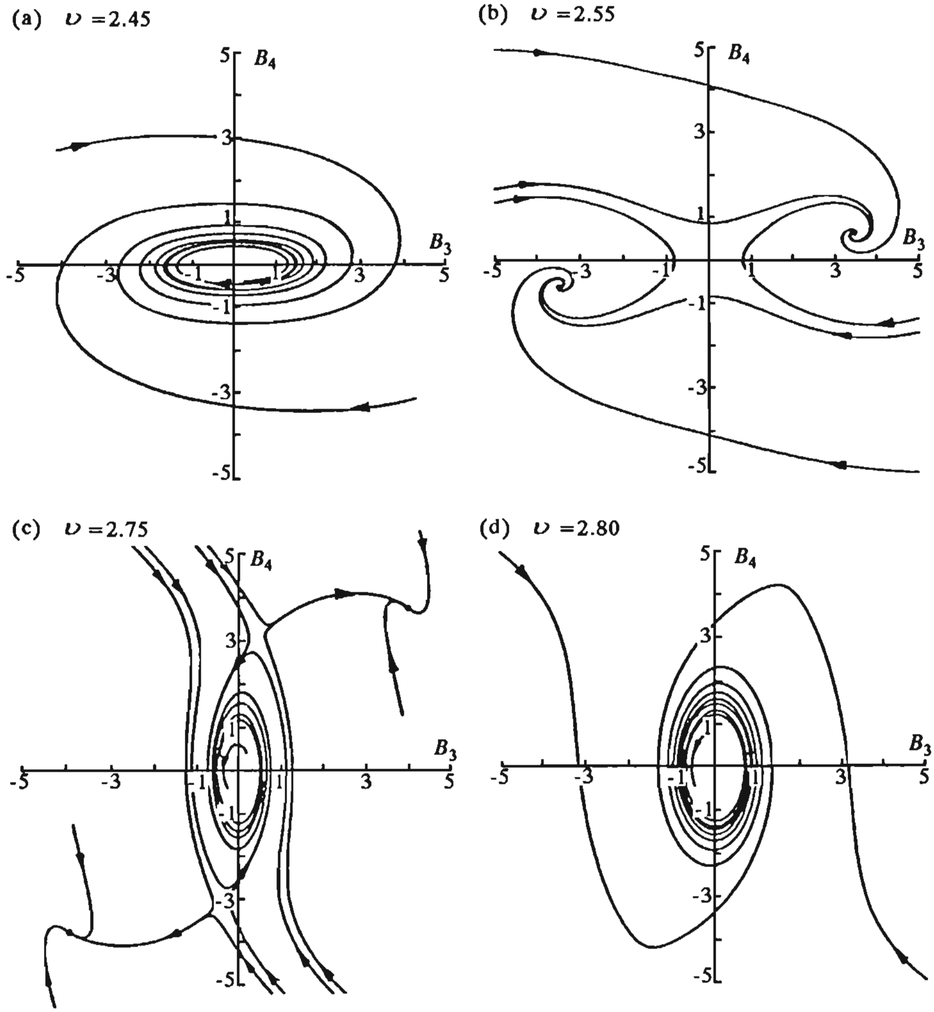

Fig. 12. Integral curves on Hayashi's surface; the main parametric resonance with respect to the p_2 frequency

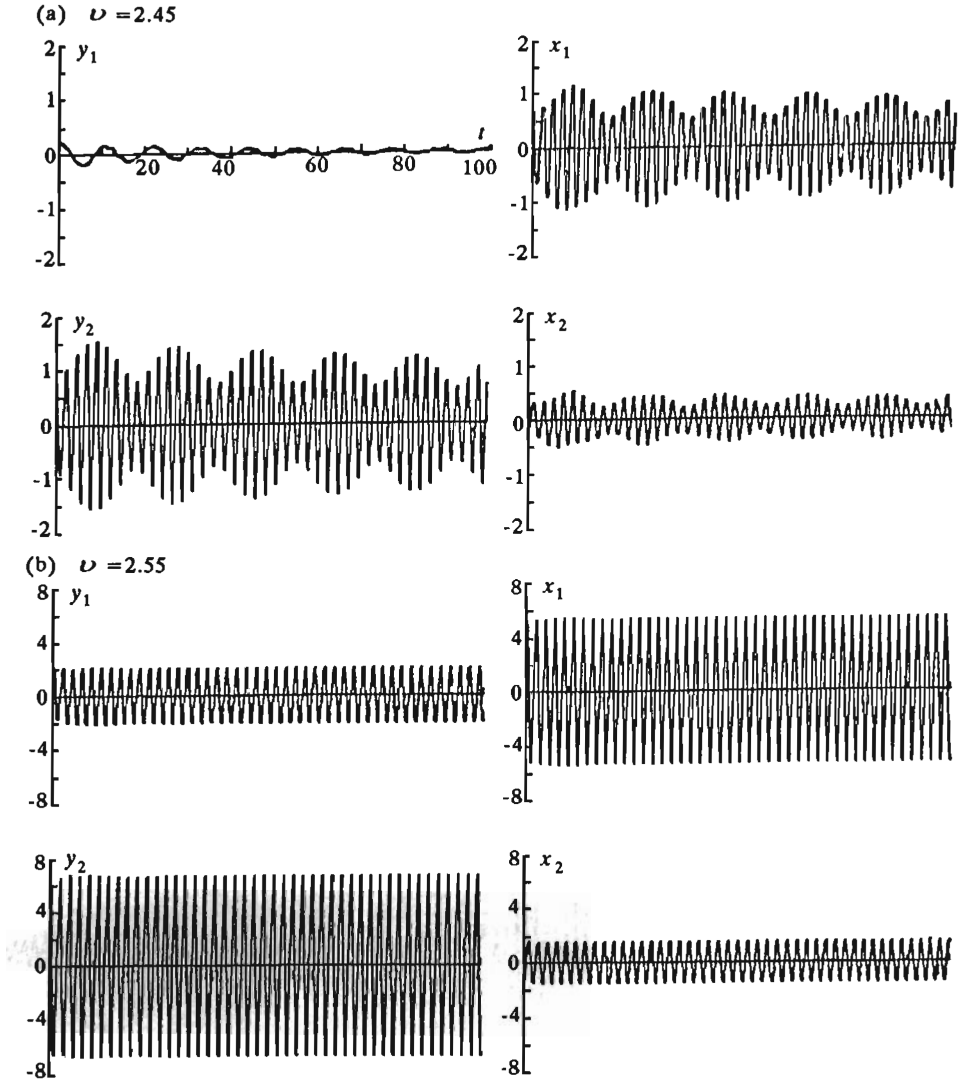

Fig. 13. Time courses in generalized and quasi-normal coordinates; resonance with respect to the frequency p_2

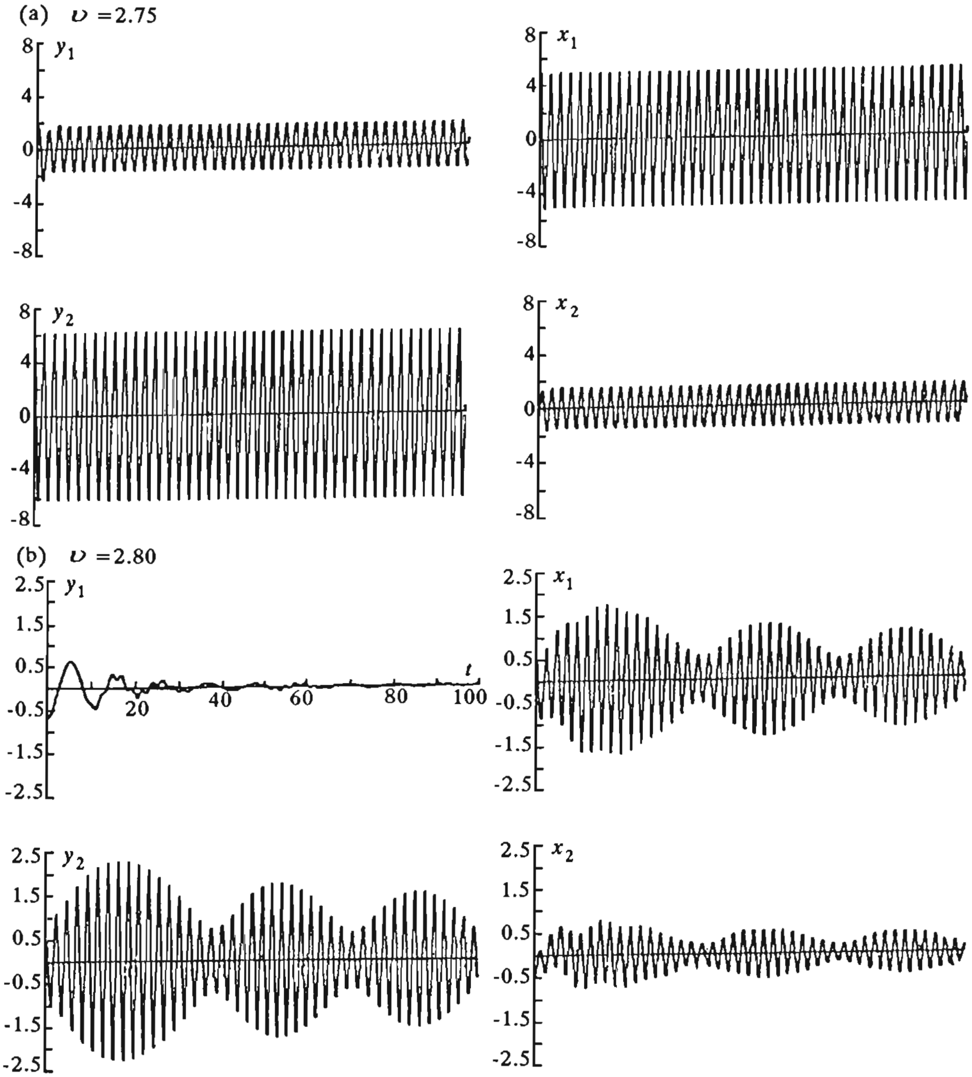

Fig. 14. Time courses in generalized and quasi-normal coordinates; resonance with respect to the frequency p_2

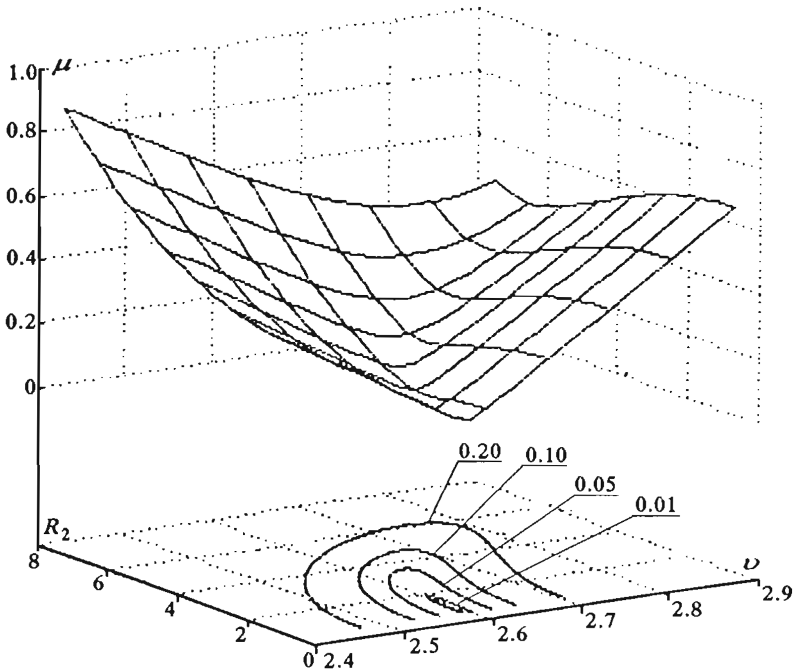

Fig. 15. Chart of $\mu(R_1, v)$ for the p_1 resonance with marked lines for $\mu = 0.1,\ 0.2,\ 0.3,\ 0.4$

$v = 0.55 \div 0.57$ area the vibrations synchronization phenomenon appears. In these frequencies range occur only frequency vibrations (with the v frequency), which are reduced to one marmal coordinate y_1. The resonance curve lies above v axis and has a closed form. Outside the synchronization area the system motion is characterized by the almost periodic vibrations. There are dominant self-excited vibrations in this area, represented by stable boundary cycles in Hayashi's surface. In the neighbourhood of free vibrations second frequency p_2, the vibrations synchronization appears in significantly wider area. In the frequency range $v = 2.47 \div 2.80$ the self-excited vibrations are pulled in by the parametric vibrations. The system motion reveals in the form of quasi-normal coordinate y_2. Outside the synchronization area almost periodical vibrations appear, represented by the boundary cycles on the phase surface. The phase curve obtained is characterized by the vibration amplitudes significantly greater than in case of the first frequency resonance. The parameter μ influence on the vibration amplitudes and synchronization area width has been detected, as well (Fig.8 and Fig.15). In both resonance states the parameter μ growth results in the vibrations amplitude growth and the synchronization area extension. The analytical results are of good qualitative agreement with analog and computer simulation results.

However, there apeared quantitative differences. In the p_1 frequency neighbourhood the vibrations amplitudes calculated analytically on the ground of first approximation are about 5% smaller than those determined numerically, in the p_2 frequency neighbourhood this difference amounts about 15%. The synchronization area width calculated analytically is narrowed in comparison to analog and computer results.

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Drgania nieliniowego układu parametryczno-samowzbudnego o dwóch stopniach swobody

Streszczenie

W pracy zbadano drgania nieliniowego ukladu o dwóch stopniach swobody, w którym występuje oddzialywanie drgań parametrycznych i samowzbudnych. Analizę przeprowadzono dla glównego rezonansu parametrycznego w otoczeniu pierwszej i drugiej częstości drgań wlasnych. Wyznaczono analitycznie amplitudy drgań ukladu oraz szerokości obszarów synchronizacji. Zbadano stateczność otrzymanych rozwiązań okresowych. Badania analityczne zweryfikowano i uzupełniono wynikami symulacji cyfrowej i modelowania analogowego.

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