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ASYMPTOTIC HOMOGENIZATION METHOD FOR COUPLED FIELDS IN PERIODICALLY HETEROGENEOUS DEFORMABLE SOLIDS

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It is shown that the procedure of homogenization can be applied to determination of macroscopic constitutive relations of a magnetoelastic and a ferromagnetoelastic periodically heterogeneous media. The effective material coefficients of a layered ferromagnetic structure are calculated.

1. Introduction

During the last two decades an increasing amount of research has been conducted to develop methods and procedures for improving description of macroproperties for given microinhomogeneous structure of solid media. In the case of different physical fields interaction in solids the problem of macrodescription is especially interesting from theoretical and experimental points of view. The spectrum of physical phenomena in coupled fields is discussed e.g. by Maugin (1988), Nowacki (1983), Parkus (1972). The method used in this paper, called homogenization, Bensoussan et al. (1978), consists in replacing the model of heterogeneous medium with a periodic structure by an equivalent model which is homogeneous. Equivalence is understood in the sense that the solution to the initial-boundary value problem under consideration for a periodically heterogeneous body is "close" to the solution to the related initial-boundary value problem for the equivalent homogeneous body, effective coefficients of which are constant. Homogenization was applied previously by many authors to calculation of macro-behaviour of thermoelastic field, piezoelectricity in solids, Gałka et al. (1992), a perfect conducting solid, Bytner and Gambin (1992), and many others. The example of a practical meaning of the method was shown in Mirgaux and Saint Jean Paulin (1982), where superconducting multifilamentary composites in presence of a weak electromagnetic field is studied. The macroscopic transverse conductivity describing a loss of energy dissipated in a matrix (with fibres as superconductors) is in agreement with experimental data. Eddy-current non-destructive tests for electromagnetoelastic materials are of more practical interest in the homogenization technique. Zhou and Hsieh (1988) based the theoretical modelling of a composite structure on the model similar to that of the self-consistent model of matrix-inclusion composites and even in such a case interesting results were obtained. The method of homogenization is more promising than the self-consistent scheme if 2-dimensional examples are calculated. Among various approaches of homogenization theory the variational Γ-convergence method e.g. Telega (1991), the Bloch expansion method Maugin and Turbé (1991) and the asymptotic technique e.g. Galka et. al. (1992) can be mentioned, the latter being applied in this paper.

Our goal is to compare the macroscopic behaviour of a micro-heterogeneous magnetoelastic and a ferromagnetoelastic solid. By applying the theory of homogenization the method of the two-scale asymptotic expansion is exploited. As a result the homogenized system of field equations and constitutive relations are obtained. All formulae include the solutions to so called "problems on the cell". The semigroup theory is used to derive the effective electric conductivity and effective dielectric constants for the case of magnetoelasticity. The integro-differential operator appears (similarly as in the homogenization procedure applied to the viscoelasticity, cf Sanchez-Palencia (1980)) in contrast to the differential homogenized laws for the case of ferromagnetoelasticity. The global constitutive laws are analytically calculated in the case of a ferromagnetoelastic layered composite.

Denotation

$\boldsymbol{b}$ – magnetic induction
$\boldsymbol{B}_0$ – initial magnetic induction
c – light velocity
$\mathbf{c}$ – elastic Hooke tensor
$\boldsymbol{D}$ – electric induction
$\boldsymbol{E}$ – electric field

$\boldsymbol{H}_0$ – initial magnetic field
$\boldsymbol{h}$ – magnetic field
$\boldsymbol{j}$ – electric current
$\boldsymbol{M}$ – magnetization vector
$\boldsymbol{P}$ – body forces
$\boldsymbol{u}$ – elastic displacement
χ – magnetic susceptibility
$\boldsymbol{\epsilon}$ – electric permeability
$\boldsymbol{\mu}$ – magnetic permeability
$\boldsymbol{\eta}$ – electric conductivity
ρ – mass density
ρ_0 – density of charges
the dot over denotes time derivative, $\times$ denotes cross product.

2. Basic equations of magnetoelasticity

The periodically heterogeneous, linear elastic solid with a finite electric conductivity is considered. Within the framework of the fenomenological theory for "slowly" moving elastic bodies, i.e., with the relativistic effects neglected, such a physical problem is governed by Maxwell equations, equations of motion and constitutive relations with the appropriate couplings between fields.

On the assumption of a strong initial magnetic induction $\boldsymbol{B}_0$ and small deformations only a slight change in the magnetic field vector $\boldsymbol{H}$ with respect to the primary field $\boldsymbol{H}_0$ occurs. In this connection it can be written

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h} \qquad\qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$$

where $\boldsymbol{h}$ and $\boldsymbol{b}$ are sufficiently small increments of pertinent fields.

Using the above relation and disregarding all products of small magnitudes, i.e. of an order higher than linear, the following set of equations is obtained by Nowacki (1983)

— Maxwell equations

$$\begin{aligned} \mathrm{rot}\boldsymbol{h} &= \boldsymbol{j} + \dot{\boldsymbol{D}} &\qquad \mathrm{div}\boldsymbol{b} &= 0 \\ \mathrm{rot}\boldsymbol{E} &= -\dot{\boldsymbol{b}} &\qquad \mathrm{div}\boldsymbol{D} &= \rho_e \end{aligned} \tag{2.1}$$

— equations of motion

$$\rho\ddot{u} = \mathrm{divcgrad}\boldsymbol{u} + \boldsymbol{j} \times \boldsymbol{B}_0 + \boldsymbol{P} \tag{2.2}$$

— constitutive relations

$$
\begin{aligned}
&\boldsymbol{j} = \boldsymbol{\eta}\boldsymbol{E} + \boldsymbol{\eta}(\dot{\boldsymbol{u}} \times \boldsymbol{B}_0) \qquad\qquad \boldsymbol{b} = \boldsymbol{\mu}\boldsymbol{h} \\
&\boldsymbol{D} = \boldsymbol{\epsilon}[\boldsymbol{E} + (\dot{\boldsymbol{u}} \times \boldsymbol{B}_0)] - \frac{1}{c^2}(\dot{\boldsymbol{u}} \times \boldsymbol{H}_0)
\end{aligned}
\tag{2.3}
$$

The above equations are valid in the domain B in the space E^3 occupied by the body in its natural state. The traces of fields $\boldsymbol{u}$, $\boldsymbol{h}$, $\boldsymbol{E}$ are assumed to be known on the boundary of the domain B. Besides the homogeneous initial conditions are assumed. It is the case when the motion of the body is caused by forces $\boldsymbol{P}$ acting inside.

The heterogeneous structure of the medium is given by the Y-periodic tensor functions of variable $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$ (ε is a small parameter representing the size of inhomogeneity) satisfying the known conditions (cf Nowacki (1983))

$$
\begin{aligned}
&\mu_{ij} = \mu_{ji} && \mu_{ij}\xi_i\xi_j \geq \gamma|\xi|^2 \\
&\eta_{ij} = \eta_{ji} && \eta_{ij}\xi_i\xi_j \geq \delta|\xi|^2 \\
&\epsilon_{ij} = \epsilon_{ji} && \epsilon_{ij}\xi_i\xi_j \geq \varphi|\xi|^2 \\
&c_{ijkr} = c_{jikr} = c_{krij} && c_{ijkr}\zeta_{ij}\zeta_{kr} \geq \psi|\zeta|^2
\end{aligned}
\tag{2.4}
$$

for all $\boldsymbol{\xi} \in \mathcal{R}^3$ and all $\boldsymbol{\zeta} \in (\mathcal{R}^3 \times \mathcal{R}^3)_{sym}$, where γ, δ, φ, ψ are positive constants, and Y denotes the unit cell.

3. Effective constitutive relations

It is assumed that the solutions to (2.2) and (2.3) have the form of a two scale asymptotic expansion

$$
\begin{aligned}
\boldsymbol{h}^\varepsilon(\boldsymbol{x},t) &= \boldsymbol{h}^0(\boldsymbol{x},\boldsymbol{y},t) + \varepsilon\boldsymbol{h}^1(\boldsymbol{x},\boldsymbol{y},t) + \dots \\
\boldsymbol{u}^\varepsilon(\boldsymbol{x},t) &= \boldsymbol{u}^0(\boldsymbol{x},\boldsymbol{y},t) + \varepsilon\boldsymbol{u}^1(\boldsymbol{x},\boldsymbol{y},t) + \dots \\
\boldsymbol{E}^\varepsilon(\boldsymbol{x},t) &= \boldsymbol{E}^0(\boldsymbol{x},\boldsymbol{y},t) + \varepsilon\boldsymbol{E}^1(\boldsymbol{x},\boldsymbol{y},t) + \dots
\end{aligned}
\tag{3.1}
$$

where $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$ and $\boldsymbol{h}^i$, $\boldsymbol{u}^i$, $\boldsymbol{E}^i$ are Y-periodic functions. In the sequel a coordinate t will be postponed.

Substituting Eqs (3.1) into (2.1), (2.2), (2.3) and bearing in mind

$$
\frac{\partial f^\varepsilon(\boldsymbol{x})}{\partial x_i} = \frac{\partial f(\boldsymbol{x},\boldsymbol{y})}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial f(\boldsymbol{x},\boldsymbol{y})}{\partial y_i} \qquad\qquad \boldsymbol{y} = \frac{\boldsymbol{x}}{\varepsilon}
$$

for $f^\varepsilon\left(\boldsymbol{x}\right) = f(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon})$ the hierarchy of equations can be obtained by equating coefficients of the same power of ε.

- Equating terms of ε^{-2} we obtain that $\boldsymbol{u}^0$ is independent of $\boldsymbol{y}$, i.e., $\boldsymbol{u}^0(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{u}^0(\boldsymbol{x})$
- Equating terms of ε^{-1} we obtain that $\boldsymbol{u}^1(\boldsymbol{x}, \boldsymbol{y}) = -\chi(\boldsymbol{y})\mathrm{grad}_x \boldsymbol{u}^0(\boldsymbol{x})$, where $\chi(\boldsymbol{y})$ is a solution to the unit cell problem

$$\mathrm{div}_y[\mathsf{c}(\boldsymbol{y})\mathrm{grad}_y \chi(\boldsymbol{y})] = \mathrm{div}_y \mathsf{c}(\boldsymbol{y}) \tag{3.2}$$

We defined the operator mean value $< \cdot >$ on any Y-periodic function $f(\cdot, \boldsymbol{y})$ where

$$< f(\cdot, \boldsymbol{y}) >= \frac{1}{|Y|} \int_Y f(\cdot, \boldsymbol{y})\, dy$$

Besides $\boldsymbol{E}^0(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{h}^0(\boldsymbol{x}, \boldsymbol{y})$ are gradients in $\boldsymbol{y}$ ($\boldsymbol{x}$ is a parameter).
It means that

$$\begin{aligned} \boldsymbol{E}^0(\boldsymbol{x}, \boldsymbol{y}) - < \boldsymbol{E}^0(\boldsymbol{x}, \boldsymbol{y}) > &= \mathrm{grad}_y \Phi(\boldsymbol{x}, \boldsymbol{y}) \\ \boldsymbol{h}^0(\boldsymbol{x}, \boldsymbol{y}) - < \boldsymbol{h}^0(\boldsymbol{x}, \boldsymbol{y}) > &= \mathrm{grad}_y \Psi(\boldsymbol{x}, \boldsymbol{y}) \end{aligned} \tag{3.3}$$

Functions Φ and Ψ are Y-periodic in $\boldsymbol{y}$.

- Equating terms of ε^0 and taking the mean value $< \cdot >$ in derived equations we obtain

$$\begin{aligned} &\mathrm{rot}_x < \boldsymbol{h}^0(\boldsymbol{x}, \boldsymbol{y}) >=< \epsilon(y)\dot{\boldsymbol{E}}^0(\boldsymbol{x}, \boldsymbol{y}) > + < \epsilon(y) > \left(\ddot{\boldsymbol{u}}^0(\boldsymbol{x}) \times \boldsymbol{B}_0\right) + \\ &\quad - \frac{1}{c^2}\left(\ddot{\boldsymbol{u}}^0(\boldsymbol{x}) \times < \boldsymbol{H}_0(y) >\right) + < \eta(y)\boldsymbol{E}^0(\boldsymbol{x}, \boldsymbol{y}) > + \\ &\quad + < \eta(y) > \left(\dot{\boldsymbol{u}}^0(\boldsymbol{x}) \times \boldsymbol{B}_0\right) \\ &\mathrm{rot}_x < \boldsymbol{E}^0(\boldsymbol{x}, \boldsymbol{y}) >= - < \mu(y)\dot{\boldsymbol{h}}^0(\boldsymbol{x}, \boldsymbol{y}) > \\ &< \rho > \ddot{\boldsymbol{u}}^0(\boldsymbol{x}) = \mathrm{div}_x \mathsf{c}^h \mathrm{grad}_x \boldsymbol{u}^0(\boldsymbol{x}) + \\ &\quad + \left[< \eta(y)\boldsymbol{E}^0(\boldsymbol{x}, \boldsymbol{y}) > + < \eta(y) > \left(\dot{\boldsymbol{u}}^0(\boldsymbol{x}) \times \boldsymbol{B}_0\right)\right] \times \boldsymbol{B}_0 + \boldsymbol{P} \end{aligned} \tag{3.4}$$

where the constant homogenized tensor of elasticity is defined by

$$\mathsf{c}^h =< \mathsf{c}(y) - \mathsf{c}(y)\mathrm{grad}_y \chi(y) > \tag{3.5}$$

In what follows the abbreviate notation will be used omitting arguments where possible.

Now the homogenized constitutive laws have to be obtained. First we look for the homogenized relation for magnetic induction and magnetic field. Taking the divergence of Eq (2.1), using Eqs (3.1) and (3.3) and averaging over the cell Y we obtain

$$\mathrm{div}_y\Big[\boldsymbol{\mu}(\mathrm{grad}_y\Psi + <\boldsymbol{h}^0>)\Big] = 0 \tag{3.6}$$

Eq (3.6) implies

$$\Psi(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{\chi}^1(\boldsymbol{y}) <\boldsymbol{h}^0(\boldsymbol{x},\boldsymbol{y})> \tag{3.7}$$

where $\boldsymbol{\chi}^1(\boldsymbol{y})$ is a solution of equation on the cell

$$\mathrm{div}_y\Big[\boldsymbol{\mu}(\boldsymbol{y})\mathrm{grad}_y\boldsymbol{\chi}^1(\boldsymbol{y})\Big] = \mathrm{div}_y\boldsymbol{\mu}(\boldsymbol{y}) \tag{3.8}$$

Then the mean values of $\boldsymbol{h}^0$ and $\boldsymbol{b}^0$ are related by

$$<\boldsymbol{b}^0>= \boldsymbol{\mu}^h <\boldsymbol{h}^0> \tag{3.9}$$

where the homogenized constant tensor $\boldsymbol{\mu}^h$

$$\boldsymbol{\mu}^h =<\boldsymbol{\mu}(\boldsymbol{y}) - \boldsymbol{\mu}(\boldsymbol{y})\mathrm{grad}_y\boldsymbol{\chi}^1(\boldsymbol{y})> \tag{3.10}$$

We conclude that the homogenized magnetic permeability has the similar form as the elastic homogenized tensor.

Now we look for homogenized relations for electric induction and electric current.

Taking the divergence of Eq (2.1), using the asymptotic expansion (3.1) and (3.3) and averaging over the cell Y we obtain

$$\begin{gathered}\mathrm{div}_y\Big[\Big(\epsilon\frac{\partial}{\partial t} + \eta\Big)\Big(<\boldsymbol{E}^0> + \mathrm{grad}_y\Phi\Big)\Big] + \mathrm{div}_y\Big[\epsilon(\ddot{\boldsymbol{u}}^0 \times \boldsymbol{B}_0)\Big] + \\ -\frac{1}{c^2}\mathrm{div}_y(\ddot{\boldsymbol{u}}^0 \times \boldsymbol{H}_0) + \mathrm{div}_y\Big[\eta(\dot{\boldsymbol{u}}^0 \times \boldsymbol{B}_0)\Big] = 0\end{gathered} \tag{3.11}$$

Eq (3.11) contains the time derivative of $\boldsymbol{E}^0$ and it is solved by Bytner and Gambin (1993) using the semigroup theory.

To show the final formulae for homogenized relations let us introduce

$$\tilde{V} = \Big\{\Theta \;:\; \Theta \in H^1_{loc}(\mathcal{R}^3),\; Y\text{-per.},\; <\Theta>= 0\Big\} \tag{3.12}$$

equipped with the scalar product

$$(\Phi, \Theta)_{\tilde{V}} = \int_Y \epsilon_{ij}(y) \frac{\partial \Phi}{\partial y_i} \frac{\partial \Theta}{\partial y_j} \, dy \tag{3.13}$$

the operator A (which is bounded and symmetric from $\tilde{V}$ into itself) and the elements of $\tilde{V}$: f_j^1, f_j^2, f_j^3, f_j^4, $j = 1,2,3$ defined by

$$\begin{aligned}
&(A\Phi, \Theta)_{\tilde{V}} = \int_Y \eta_{ij} \frac{\partial \Phi}{\partial y_i} \frac{\partial \Theta}{\partial y_j} \, dy && (f_j^1, \Theta)_{\tilde{V}} = \int_Y \epsilon_{ij} \frac{\partial \Theta}{\partial y_i} \, dy \\
&(f_j^2, \Theta)_{\tilde{V}} = \int_Y \eta_{ij} \frac{\partial \Theta}{\partial y_i} \, dy && (f_j^4, \Theta)_{\tilde{V}} = \int_Y \mu_{ij}^{-1} \frac{\partial \Theta}{\partial y_i} \, dy \\
&f_j^3 = A f_j^1 - f_j^2
\end{aligned} \tag{3.14}$$

Finally the macroscopic constitutive relations have the form

$$\begin{aligned}
< D_i^0 > \; &= \; b_{ij}^\epsilon < E_j^0 > + \int_0^t d_{ij}^\epsilon < E_j^0 > \, ds + b_{ij}^\epsilon (v^0 \times B_0)_j + \\
&+ \int_0^t d_{ij}^\epsilon (v^0 \times B_0)_j \, ds - \frac{1}{c^2} < \mu_{rm}^{-1} > \varepsilon_{ikr} v_k^0 B_{0m} + \\
&+ P_{ij}^\epsilon \varepsilon_{jnr} v_n^0 B_{0r} - \frac{1}{c^2} \int_0^t q_{ij}^\epsilon \varepsilon_{jnr} v_n^0 B_{0r} \, ds + \\
< j_i^0 > \; &= \; b_{ij}^\eta < E_j^0 > + \int_0^t d_{ij}^\eta < E_j^0 > \, ds + b_{ij}^\eta (v^0 \times B_0)_j + \\
&+ \int_0^t d_{ij}^\eta (v^0 \times B_0)_j \, ds - \frac{1}{c^2} p_{ij}^\eta \varepsilon_{jnr} v_n^0 B_{0r} - \frac{1}{c^2} \int_0^t q_{ij}^\eta \varepsilon_{jnr} v_n^0 B_{0r} \, ds
\end{aligned} \tag{3.15}$$

where the homogenized coefficients have the following representations

$$\begin{aligned}
&b_{ij}^\epsilon = < \epsilon_{ij} - \epsilon_{ik} \partial_k f_j^1 > && b_{ij}^\eta = < \eta_{ij} - \eta_{ik} \partial_k f_j^1 > \\
&d_{ij}^\epsilon = < \epsilon_{ik} \partial_k f_j^3 e^{-A\xi} > && d_{ij}^\eta = < \eta_{ik} \partial_k f_j^3 e^{-A\xi} > \\
&p_{ij}^\epsilon = < \epsilon_{ik} \partial_k f_j^4 > && p_{ij}^\eta = < \eta_{ik} \partial_k f_j^4 > \\
&q_{ij}^\epsilon = < \epsilon_{ik} \partial_k A f_j^4 e^{-A\xi} > && q_{ij}^\eta = < \eta_{ik} \partial_k A f_j^4 e^{-A\xi} >
\end{aligned} \tag{3.16}$$

The homogenized set of Eqs (3.5) and (3.15) interrelates macrofields $\boldsymbol{u}^0$, $<\boldsymbol{E}^0>$, $<\boldsymbol{h}^0>$ with $<\boldsymbol{j}^0>$, $<\boldsymbol{D}^0>$, $<\boldsymbol{b}^0>$. It has to be emphasized that the expressions (3.15) contain the integro-differential terms. The formulae (3.16) can be calculated for spacial cases only, becouse of complicated form ($e^{-A\xi}$ is a infinite sum of operators). The one dimensional example is calculated by Bytner and Gambin (1993). The initial static field $\boldsymbol{H}_0$ is generated in such a way, that the constant static magnetic induction $\boldsymbol{B}_0$ inside the body is produced, being responsible for the restrictions on the type of heterogeneities which can be treated by the method applied. Namely, taking into account that $\mathrm{rot}\boldsymbol{H}_0 = \boldsymbol{0}$ and $\boldsymbol{H}_0 = \boldsymbol{\mu}^{-1}(x)\boldsymbol{B}_0$ the following conditions imposed on the tensor field $\boldsymbol{\mu}^{-1}(x)$ must be fulfilled

$$\varepsilon_{ijk}\partial_j\mu^{-1}_{kr}(x)B_{0r} = 0$$

One should take into account the above conditions when numerical calculations have to be done.

4. Homogenization of a ferromagnetoelastic solid

Let us assume about a magnetic material that:

- The magnetostrictive and piezostrictive effects are neglected
- The elastic material is linear
- The electromagnetoelastic material is a perfect conductor (electric free charges and electric displacement current may be neglected)
- The velocity field of a material is small as compared to the light velocity so that relativistic effects are negligible
- The magnetic material is itself linear (so called "soft" ferromagnetic material), i.e.

$$\boldsymbol{M} = \chi \boldsymbol{H} \qquad \boldsymbol{B} = \boldsymbol{\mu}_0(1+\chi)\boldsymbol{H}$$

Under the above conditions the governing field equations are given by a set of nonlinear coupled partial differential equations of Moon (1984). On the extra assumption

$$\boldsymbol{H}(\boldsymbol{x},t) = \boldsymbol{H}^0 + \boldsymbol{h}(\boldsymbol{x},t) \qquad |\boldsymbol{H}^0| \ll |\boldsymbol{h}|$$

these equations can be linearized cf Zhou and Hsieh (1988). They include
— Maxwell equations

$$\begin{aligned} &\mathrm{rot}\boldsymbol{h} = \boldsymbol{j} \qquad\qquad \mathrm{div}\boldsymbol{b} = 0 \\ &\mathrm{rot}\boldsymbol{E} = -\dot{\boldsymbol{b}} \end{aligned} \tag{4.1}$$

— equations of motion

$$\begin{aligned} \rho\ddot{\boldsymbol{u}} &= \mathrm{div}(\mathbf{c}\,\mathrm{grad}\boldsymbol{u}) + \mathrm{div}\Big[\mu\chi(\boldsymbol{H}^0\boldsymbol{h} + \boldsymbol{h}\boldsymbol{H}^0)\Big] + (\boldsymbol{j}\times\boldsymbol{B}^0) + \\ &+ \mu\chi\Big[(\boldsymbol{H}^0\cdot\boldsymbol{D})\boldsymbol{h} + (\boldsymbol{h})\cdot\boldsymbol{D})\boldsymbol{H}^0\Big] \end{aligned} \tag{4.2}$$

— relations for $\boldsymbol{j}$ and $\boldsymbol{b}$

$$\boldsymbol{j} = \eta(\boldsymbol{E} + \dot{\boldsymbol{u}}\times\boldsymbol{B}^0) \qquad\qquad \boldsymbol{b} = \mu\boldsymbol{h} \tag{4.3}$$

and the appropriate boundary conditions.

The heterogeneous structure of the medium is caused by η and $\mathbf{c}$ being the Y-periodic tensor functions of variable $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$ and satisfying the conditions (2.4).

The magnetic permeability and magnetic susceptibility are assumed to be constant. This assumption simplifies the considerations.

Applying the method of two-scale asymptotic expansions it is assumed that the solutions to Eqs (4.1) and (4.2) have the form of Eq (3.1).

Like in the magnetoelastic case, $\boldsymbol{u}_0$ does not depend on $\boldsymbol{y}$, i.e. $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x},t)$. Similary, $\boldsymbol{h}_0$ is independent of $\boldsymbol{y}$.

Finally, after making necessary calculations, the homogenized set of equations for the ferromagnetoelastic medium takes the form

$$\begin{aligned} &\mathrm{rot}\boldsymbol{h}_0 = <\boldsymbol{j}_0> \\ &\mathrm{rot} <\boldsymbol{E}_0> = -\mu\dot{\boldsymbol{h}}_0 \\ &\mathrm{div}\mu\boldsymbol{h}_0 = 0 \\ &<\rho>\ddot{\boldsymbol{u}}_0 = \mathrm{div}_x\mathbf{c}^h\mathrm{grad}_x\boldsymbol{u}_0 + \mathrm{div}_x\Big[\mu_0\chi(\boldsymbol{H}^0\boldsymbol{h}_0 + \boldsymbol{h}_0\boldsymbol{H}^0)\Big] + \\ &\qquad +(<\boldsymbol{j}_0>\times\boldsymbol{B}^0) + \mu_0\chi\Big[(\boldsymbol{H}^0\cdot\nabla_x)\boldsymbol{h}_0 + (\boldsymbol{h}_0\cdot\nabla_x)\boldsymbol{H}^0\Big] \\ &<\boldsymbol{j}_0> = \eta^h\Big[<\boldsymbol{E}_0> +(\dot{\boldsymbol{u}}_0\times\boldsymbol{B}^0)\Big] \end{aligned} \tag{4.4}$$

where $\mathbf{c}^h$ is given by Eq (3.5), $\boldsymbol{\eta}^h$ has the form

$$\boldsymbol{\eta}^h = <\boldsymbol{\eta}> - <\boldsymbol{\eta} \operatorname{grad}_y \Gamma_1(\boldsymbol{y})> \tag{4.5}$$

where $\Gamma_1(\boldsymbol{y})$ is a solution to equation on a unit cell

$$\operatorname{div}_y \Big[\boldsymbol{\eta}(\boldsymbol{y}) \operatorname{grad}_y \Gamma_1(\boldsymbol{y})\Big] = \operatorname{div}_y \boldsymbol{\eta}(\boldsymbol{y}) \tag{4.6}$$

The set of Eqs (4.4) with the formulae (3.5), (4.5) and (4.6) describe the macroscopic behaviour of the special case of inhomogeneous ferromagnetoelastic body defined in this section.

The effective elastic tensor $\mathbf{c}^h$ and the effective tensor of electric conductivity $\boldsymbol{\eta}^h$ have to be calculated after a particular geometry of periodic structure is assumed. The example of layered structure is given in the next section.

5. One-dimensional example of a layered ferromagnetic medium

We assume that the medium has a layered structure and a unit cell consists of two different homogeneous but arbitrary anisotropic layers.

Define

$$\boldsymbol{y} = (y_1, y_2, y_3) \qquad\qquad y_2 = y$$

$$[\![\cdot]\!] = (\cdot)^{(1)} - (\cdot)^{(2)}$$

$$\kappa(y) = \begin{cases} 1 & \text{if } y \text{ belongs to a material with properties (1)} \\ 0 & \text{if } y \text{ belongs to a material with properties (2)} \end{cases}$$

$<\kappa(y)> = \xi$ volume fraction of material (1)

$(1-\xi)$ volume fraction of material (2)

and assume that the material coefficients have the form

$$\mathbf{c}(\boldsymbol{y}) = \kappa(y)[\![\mathbf{c}]\!] + \mathbf{c}^{(2)} \qquad\qquad \boldsymbol{\mu} = \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)}$$

$$\boldsymbol{\eta}(\boldsymbol{y}) = \kappa(y)[\![\boldsymbol{\eta}]\!] + \boldsymbol{\eta}^{(2)} \qquad\qquad \chi = \chi^{(1)} = \chi^{(2)}$$

In order to obtain $\mathbf{c}^h$ Eq (3.5) has to be used and on the assumptions made above we get

$$c^h_{ijkl} = < c_{ijkl} - c_{ij2n} \frac{\partial \Gamma^{kl}_n}{\partial y} >$$

where components of Γ_n^{kl} fulfil the equations

$$\frac{\partial c_{i22n}}{\partial y}\frac{\partial \Gamma_n^{kl}}{\partial y} = \frac{\partial c_{i2kl}}{\partial y}$$

Integrating both sides of the above equations we have

$$c_{i22n}\frac{\partial \Gamma_n^{kl}}{\partial y} = c_{i2kl} + s_{ikl}$$

s_{ikl} =*constants*, being determined by the uniqueness condition of solutions Γ_n^{kl} in the class of periodic functions

$$s_{ikl} = -< c_{i22n}^{-1} >^{-1} < c_{j22n}^{-1} c_{j2kl} >$$

Thus

$$\begin{aligned} c_{ijkl}^h &= < c_{ijkl} > - < c_{ij2n} c_{p22n}^{-1} c_{p2kl} > + \\ &+ < c_{ij2n} c_{p22n}^{-1} > < c_{p22s}^{-1} > < c_{r22s}^{-1} c_{rskl} > \end{aligned}$$

Finally, after calculations we get

$$c_{ijkl}^h = < c_{ijkl} > -(L^{-1})_{pi} [\![\mathbf{c}_{kl2i}]\!] [\![\mathbf{c}_{ij2p}]\!] \xi (1-\xi)$$

where

$$L_{pi} = c_{i2p2}^{(1)}(1-\xi) + c_{i2p2}^{(2)}\xi$$

Analogically we calculate $\boldsymbol{\eta}^h$ from Eq (4.5)

$$\eta_{ij}^h = < \eta_{ij} > - \frac{\xi(1-\xi)}{\eta_{22}^{(1)}(1-\xi) + \eta_{22}^{(2)}\xi} [\![\eta_{2i}]\!] [\![\eta_{2j}]\!]$$

The expansions for c_{ijkl}^h and η_{ij}^h demonstrate the effect of macroscopic transversal anisotropy caused by the geometry of layered structure even if tensors $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, $\boldsymbol{\eta}^{(1)}$ and $\boldsymbol{\eta}^{(2)}$ are assumed to be isotropic. In general case the effective tensors are anisotropic. The graphical illustrations of the qualitative dependence on volume fraction ξ of the particular components of tensors $\mathbf{c}^h$ and $\boldsymbol{\eta}^h$ are given by Galka et al. (1994).

6. Conclusions

It has to be emphasized that the zero-order terms in ε-expansion series of the electric and magnetic fields $\boldsymbol{E}^0$ and $\boldsymbol{h}^0$ are rapidly fluctuating i.e. they depend on the microvariable $\boldsymbol{y}$ on the contrary to the pure dependence of the displacement $\boldsymbol{u}$ on the macrovariable $\boldsymbol{x}$. In the effective relations for the mean values of the current $< \boldsymbol{j}^0 >$ and the electric induction $< \boldsymbol{D}^0 >$ we obtain the integro-differential terms (corresponding to the memory effects) which describe nonlocality in time of the effective laws. In the case of a perfect conductor when $\dot{\boldsymbol{D}} = \boldsymbol{0}$ the integro-differential terms disappear in Eqs (3.15). It is proved that the kernels in integral terms decay exponentially as times goes to infinity and consequently the "memory" vanishes exponentially. In the case of the ferromagnetoelastic material considered in Section 4, Eq (4.4), have the same form as Eqs (4.1) $\div$ (4.4) the only difference is that rapidly fluctuating coefficients $\mathbf{c}(\boldsymbol{y})$ and $\eta(\boldsymbol{y})$ are replaced by constant tensors $\mathbf{c}^h$ and η^h. There are no new effects in macrobehaviour as compare to microbehaviour in contrast to the case of magnetoelastic case.

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Metoda asymptotyczna homogenizacji dla pól sprzężonych w periodycznie niejednorodnych odkształcalnych ciałach stałych

Streszczenie

Pokazano, że procedurę homogenizacji można zastosować do wyznaczenia makroskopowych związków konstytutywnych pewnych magnetosprężystych i ferromagnetosprężystych periodycznie niejednorodnych ośrodków. Obliczono efektywne współczynniki materiałowe dla warstwowej struktury ferromagnetycznej.

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