MECHANIKA TEORETYCZNA
I STOSOWANA
Journal of Theoretical
and Applied Mechanics
4, 31, 1993

MACRO-DYNAMIC OF MICRO-PERIODIC ELASTIC BEAMS[1]

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The aim of the paper is to propose a certain method of macro-modelling of micro-periodic elastic beams. The approach is based on the concepts of a non-asymptotic macro-modell proposed by Woźniak [3]. In this paper the equations of motion for a straight linear elastic micro-periodic beam has been obtained. A solutions of a dynamic eigenvalue problem and steady state harmonic vibration have been presented. The problems were analized within the engineering theory of elastic beams.

1. Primary concepts

The analysis will be restricted to the small displacement gradient theory of beams and to linear-elastic materials.

In the paper the bending vibrations of the straight beam of a finite length L with the periodic variable flexural rigidity is considered (Fig.1).

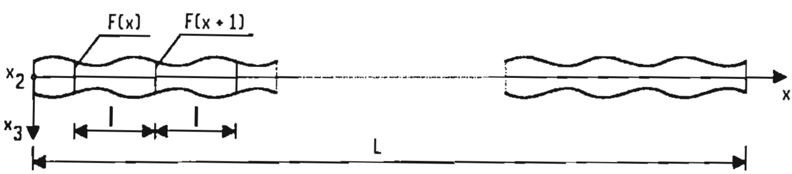

Fig. 1.

Let in the undeformed configuration the axis of a beam coincide with the intervall $[0, L]$ of the x-axis. Let X_2, X_3 – axis be principal central inertia axes of an arbitrary cross-section $F(x)$, $x \in [0, L]$. A beam has l-periodic structure, it means that $F(x) = F(x + l)$ and material constant (the Young

[1]The research was supported by the State Committee for Scientific Research Warsaw, under grant 3 3310 92 03

modulus) $E(x) = E(x+l)$ and the mass density related to the x-axis of a beam $\rho(x) = \rho(x+l)$, for every $x \in [0, L-l]$.

We assume that l is sufficiently small compared to L, $l \ll L$ and $L = Pl$, where P is any integer number.

The condition $l \ll L$ made it possible to treat the beam under consideration as the micro-periodic structure.

Throught out the paper indices a, b run over $1, 2, ..., n$, where $n \geq 1$, $m = 1, 2, ..., p = 0, 1, ..., P$. The summation convention for a, b holds.

Setting $\mathcal{L} \equiv [0, L]$ we shall refer to $\mathcal{L}$ as a representative length element of the beam.

The coordinate y, $y \in [0, l]$ will be called a micro-coordinate and $x \in [0, L]$ will denote the macro-coordinate of the periodic beam under consideration.

For any differentiable function $g(x,t)$, where t is the time coordinate, $t \in [0, \infty]$, we define

$$g' \equiv \frac{\partial g(x,t)}{\partial x} \qquad\qquad \dot{g} \equiv \frac{\partial g(x,t)}{\partial t}$$

For any integrable function $G(\cdot)$ defined on the interval $[0, L]$ the averaged value is

$$<G>(x) \equiv \frac{1}{l}\int\limits_0^l G(x+y)\,dy \qquad\qquad \text{for} \quad x \in [0, L-l]$$

2. Dynamic macro-modelling

The method presented bellow is based on refined macro-dynamics of micro-periodic composites (Woźniak, 1993 and [3]). We start our analysis with some preliminary concepts.

Let $g(\cdot)$ be an arbitrary continuous function defined on $[0, L]$. The function $g(\cdot)$ will be called a $\mathcal{L}$-macro function if $g(x) \cong g(z)$ for every x, z such that $|x - z| < l$ in the whole domain of the definition of $g(\cdot)$. A continuous function $g(\cdot)$ having continuous derivatives up to cut order will be called $\mathcal{L}$-macro function if $g(\cdot)$ and all its derivatives are $\mathcal{L}$-macro functions.

The independent real-valued functions $h_a(x)$, $a = 1, ..., n$ satysfy the following conditions

1. they are defined and continuous for every $x \in R$

2. h_a are l-periodic, $h_a(x) = h_a(x+l)$, $x \in [0, L-l]$

3. $<h_a>(x) = 0$ (2.1)

4. $<\rho h_a>(x) = 0$

Functions $h_a(\cdot)$ characterize micro-oscillations of a beam due to its micro-periodic structure and will be called micro-shape functions.

We postulate that

- a displacement field along the x-axis and towards the X_3-axis $W(\cdot,t)$ for every $x \in [0,L]$ can be represented by

$$w(x,t) = W(x,t) + h_a(x)Q^a(x,t) \tag{2.2}$$

  where $W(\cdot,t)$ and $Q^a(\cdot,t)$ are arbitrary, independent, sufficiently regular $\mathcal{L}$ - macro fields defined on $[0,L]$ and $h_a(\cdot)$ are postulated a priori micro-shape functions.

We shall also postulate that

- in derivatives of the displacement $w(x,t)$ the terms involving functions $h_a(\cdot)$ can be neglected compared to terms involving derivatives of $h_a(\cdot)$.

Hence, we assume that

$$\begin{aligned} w'(x,t) &= W'(x,t) + h'_a(x)Q^a(x,t) \\ w''(x,t) &= W''(x,t) + h''_a(x)Q^a(x,t) \end{aligned} \tag{2.3}$$

The first from the aforementioned postulates is implied by the fact that the motion of a micro-periodic beam can be obtained by a superimposition of micro-oscillations $h_aQ^a(\cdot,t)$ on a certain fundamental motion desciribed by $\mathcal{L}$-macro function $W(\cdot,t)$.

The second postulat is motivated by the micro-ascillatory character of shape-function $h_a(\cdot)$, where $h_a \in 0(l^2)$, $h'_a \in 0(l)$, $h''_a \in 0(1)$.

For the beam under consideration the principle of virtual work will be postulated in the well known form

$$\begin{aligned} -\int_0^L M(x,t)\delta w''(x)dx &= M_0(t)\delta w'(0) - M_L(t)\delta w'(L) - T_0(t)\delta w(0) + \\ &+T_L(t)\delta w(L) + \int_0^L [f(x,t) - \rho(x)\ddot{w}(x,t)]\delta w(x)\,dx \end{aligned} \tag{2.4}$$

where

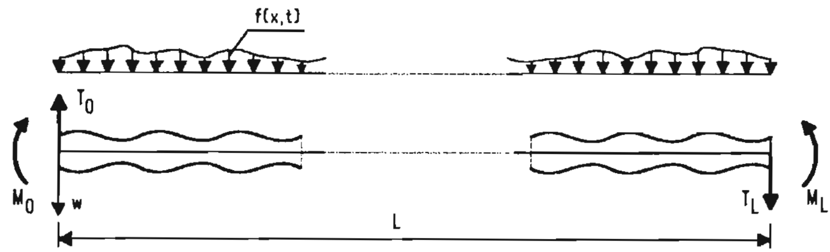

Fig. 2.

$M(x,t)$	–	bending moment at the cross-sections $x = \text{const}$
$M_0(t)$, $M_L(t)$	–	values of M at the ends of a beam
$T_0(t)$, $T_L(t)$	–	boundary values of shear forces
$f(x,t)$	–	vertical total loading distributed along the axis of a beam (see Fig.2).

The principle (2.4) is assumed to hold fold every

$$
\begin{aligned}
\delta w(x) &= \delta W(x) + h_a(x)\delta Q^a(x)\\
\delta w'(x) &= \delta W'(x) + h_a'(x)\delta Q^a(x)\\
\delta w''(x) &= \delta W''(x) + h_a''(x)\delta Q^a(x)
\end{aligned}
\tag{2.5}
$$

where $\delta W(\cdot)$, $\delta Q^a(\cdot)$ are independent functions defined on $[0, L)$ such that $\delta W(0) = \delta W(L) = 0$, $\delta Q^a(0) = \delta Q^a(L) = 0$.

It has to be emphasised that $\delta W(\cdot)$ and $\delta Q^a(\cdot)$ are $\mathcal{L}$-macro functions due to the first one from the afforementioned postulates.

Let us transform Eq (2.4) to the form which involves exclusively $\mathcal{L}$-macro functions. Following the line of approach applied by Woźniak [3] we obtain

$$
\begin{aligned}
&\int\limits_0^L M(x,t)\delta w''(x)\,dx = \int\limits_0^L M(x,t)\Big[\delta W''(x) + h_a''(x)\delta Q^a(x)\Big]\,dx \cong\\
&\cong \sum_{p=0}^{P-1}\Big[\int\limits_{pl}^{(p+1)l} M(x,t)\,dx\delta W''(pl) + \int\limits_{pl}^{(p+1)l} M(x,t)h_a''(x)\,dx\delta Q^a(pl)\Big] =\\
&= \sum_{p=0}^{P-1}\Big[<M>(pl,t)\delta W''(pl) + <Mh_a''>(pl,t)\delta Q^a(pl)\Big]l \cong\\
&\cong \int\limits_0^L \Big[\widetilde{M}(x,t)\delta W''(x) + \widetilde{M}_a(x,t)\delta Q^a(x)\Big]\,dx
\end{aligned}
\tag{2.6}
$$

where $\widetilde{M}(\cdot,t)$ and $\widetilde{M}_a(\cdot,t)$ are sufficiently regular $\mathcal{L}$-macro funtions defined on $[0,L]$ which satisfy conditions

$$\begin{aligned} \widetilde{M}(x,t) &\cong <M>(x,t) \\ \widetilde{M}_a(x,t) &\cong <Mh_a''>(x,t) \end{aligned} \tag{2.7}$$

for $x \in [0, L-l]$.

$\mathcal{L}$-macro functions $\widetilde{M}(x,t)$, $\widetilde{M}_a(x,t)$ will be refered to as macro-bending moments and hiper-bending moments, respectively.

Simiarly we also obtain

$$\begin{aligned} &\int_0^L \Big[f(x,t) - \rho(x)\ddot{w}(x,t)\Big]\delta w(x)\,dx = \\ &= \int_0^L \Big[f(x,t) - \rho(x)\ddot{W}(x,t) - \rho(x)h_b(x)\ddot{Q}^b(x,t)\Big]\cdot \\ &\cdot\Big[\delta W(x) + h_a(x)\delta Q^a(x)\Big]\,dx \cong \sum_{p=0}^{P-1}\Big[\int_{pl}^{(p+1)l} f(x,t)\,dx\delta W(pl) + \\ &+ \int_{pl}^{(p+1)l} f(x,t)h_a(x)\,dx\delta Q^a(pl) - \int_{pl}^{(p+1)l}\rho(x)\,dx\ddot{W}(pl,t)\delta W(pl) - \\ &- \int_{pl}^{(p+1)l}\rho(x)h_a(x)\,dx\ddot{W}(pl,t)\delta Q^a(pl) - \int_{pl}^{(p+1)l}\rho(x)h_b(x)\,dx\ddot{Q}^b(pl,t)\delta W(pl) - \\ &- \int_{pl}^{(p+1)l}\rho(x)h_b(x)h_a(x)\,dx\ddot{Q}^b(pl,t)\delta Q^a(pl)\Big] = \\ &= \sum_{p=0}^{P-1}\Big[<f>(pl,t)\delta W(pl) + <fh_a>(pl,t)\delta Q^a(pl) - \\ &- <\rho>(pl)\ddot{W}(pl,t)\delta W(pl) - <\rho h_a h_b>(pl)\ddot{Q}^b(pl,t)\delta Q^a(pl)\Big] \cong \\ &\cong \int_0^L \Big\{\Big[<f>(x,t) - <\rho>\ddot{W}(x,t)\Big]\delta W(x) + \\ &+\Big[<fh_a>(x,t) - <\rho h_a h_b>\ddot{Q}^b(x,t)\Big]\delta Q^a(x)\Big\}\,dx \end{aligned} \tag{2.8}$$

In formulas (2.6) and (2.8) we have taken into account both the afforementioned postulates and the properties of $\mathcal{L}$-macro functions.

On the basis of Eqs (2.4), (2.6) and (2.8) we shall introduce the following macro-approximation of the virtual work principle

$$
\begin{aligned}
&-\int\limits_0^L \Big[\widetilde{M}(x,t)\delta W''(x) + \widetilde{M}_a(x,t)\delta Q^a(x)\Big]\, dx = M_0(t)\delta W'(0) + \\
&+M_0(t)h_a'(0)\delta Q^a(0) - M_L(t)\delta W'(L) - M_L(t)h_a'(L)\delta Q^a(L) - \\
&-T_0(t)\delta W(0) - T_0(t)h_a(0)\delta Q^a(0) + T_L(t)\delta W(L) + \\
&+T_L(t)h_a(L)\delta Q^a(L) + \int\limits_0^L \Big\{\Big[<f>(x,t) - <\rho>\ddot{W}(x,t)\Big]\delta W(x) + \\
&+\Big[<fh_a>(x,t) - <\rho h_a h_b>\ddot{Q}^b(x,t)\Big]\delta Q^a(x)\Big\}\, dx
\end{aligned}
\tag{2.9}
$$

Condition (2.9) has to hold for every independent and sufficiently regular $\mathcal{L}$-macro field $\delta W(\cdot)$, $\delta Q^a(\cdot)$, such that $\delta W(0) = \delta W(L) = 0$, $\delta Q^a(0) = \delta Q^a(L) = 0$.

It can be proofed that the reduction of a virtual field $\delta W(\cdot)$, $\delta Q^a(\cdot)$ to the $\mathcal{L}$-macro field in Eq (2.9) is irrelevant.

Hence using the well known procedure we obtain

$$
\begin{aligned}
&\widetilde{M}''(x,t) + <f>(x,t) - <\rho>\ddot{W}(x,t) = 0 \\
&\widetilde{M}_a''(x,t) + <fh_a>(x,t) - <\rho h_a h_b>\ddot{Q}^b(x,t) = 0
\end{aligned}
\tag{2.10}
$$

Taking into account conditions (2.7) as well as the known interrelation

$$
M(x,t) = -B(x)w''(x,t) \tag{2.11}
$$

where $B(x)$ is the flexural rigidity of the beam, $B(x) = B(x + l)$, $x \in [0, L - l]$, and using the first of the basic postulates we conclude

$$
\begin{aligned}
&\widetilde{M}(x,t) = -<B>W''(x,t) - <Bh_b''>Q^b(x,t) \\
&\widetilde{M}_a(x,t) = -<Bh_a''>W''(x,t) - <Bh_a''h_b''>Q^b(x,t)
\end{aligned}
\tag{2.12}
$$

for $x \in (0, L)$.

Eqs (2.12) constitute the interrelation between macro-bending moments $\widetilde{M}(x,t)$, hiper-bending moments $\widetilde{M}_a(x,t)$ and macro-displacements $W(x,t)$ and correctors $Q^a(x,t)$, respectively.

Eqs (2.10) and Eqs (2.12) desribe the macro-model of a micro-periodic beam under consideration and constitute the final result of the proposed method of dynamic macro-modelling.

Let us observe that for beams with the constant cross-section $B(x) = B =$ const, under the initial conditions

$$Q^b(x,t_0)=0 \qquad \dot{Q}^b(x,t_0)=0 \qquad , \qquad x\in(0,L)$$

we obtain $Q^a(x,t)=0$, for every $x\in(0,L)$ and $t\in(t_0,t_f)$ and hence Eqs (2.10) and (2.12) reduce to the well known form. Thus we conclude that the correctors $Q^a(\cdot,t)$ describe the effect of micro-periodic beam structure on the behaviour of the beam under bending.

Substituting the RHS of Eqs (2.12) into Eqs (2.10) we obtain the following $n+1$ differential equations

$$\begin{aligned} &<B> W^{IV}(x,t)+ <Bh_b''> Q^{b''}(x,t)+ <\rho> \ddot{W}(x,t) =<f> (x,t) \\ &<Bh_a''> W''(x,t)+ <Bh_a''h_b''> Q^b(x,t)+ <\rho h_a h_b> \ddot{Q}^b(x,t) =<fh_a> (x,t) \end{aligned} \tag{2.13}$$

Eqs (2.13) describe the behaviour of linear-elastic beams with the micro-periodic structure under consideration, the inertial properties of which are described by the averaged mass density $<\rho>$ and by the micro-inertial modulae $<\rho h_a h_b>$. It has to be emphasised that the micro-intertial modulae $<\rho h_a h_b>$ depend of the small length parameter l and hence the derived equations also described the dynamic micro-behaviour of a beam. Thus, it is possible to analyse the micro-vibrations taking into account the dispersion effects.

These effects can not be shown by using asymptotic methods of homogenization (cf Bakhvalov and Panasenko (1984) and an extensive liste of references on these problems], where terms of an order $0(l)$ are neglected.

The second one from Eqs (2.13) does not involve the spatial derivatives of correctors. It follows that the boundary value problems are related to the macro-displacements $W(\cdot,t)$. At the same time the initial conditions have to be prescribed both for $W(\cdot,t)$ and $Q^a(\cdot,t)$.

The examples of applications of Eqs (2.13) will be given in Sections 3,4.

3. Eigenvalue problems

We consider the free vibrations of the straight elastic beam with the l-periodic structure. For simplicity we introduce only one micro-shape functions $h(x) \equiv h_1(x)$.

From Eqs (2.13) assuming $f(x,t)=0$ we obtain

$$\begin{aligned} &<B> W^{IV}(x,t)+ <Bh''> Q''(x,t)+ <\rho> \ddot{W}(x,t) = 0 \\ &<Bh''> W''(x,t)+ <Bh''h''> Q(x,t)+ <\rho hh> \ddot{Q} = 0 \end{aligned} \tag{3.1}$$

for $x \in (0, L)$, $t \in (t_0, t_f)$.

Eqs (3.1) have a special solution of the form

$$W(x,t) = 0$$
$$Q(x,t) = A_1 \cos \mu t + A_2 \sin \mu t$$

where A_1, A_2 are arbitrary constants and $\mu^2 = \frac{<Bh''h''>}{<\rho hh>}$. The positive constant μ is the free micro-vibration frequency.

Applying the method of separation of variables we are looking for the solution of Eqs (3.1) in the form

$$\begin{aligned} W(x,t) &= W_0(x)T(t) \\ Q(x,t) &= Q_0(x)T(t) \end{aligned} \tag{3.2}$$

Hence we obtain the equation for $T(\cdot)$ in the form

$$\ddot{T}(t) + \omega^2 T(t) = 0 \tag{3.3}$$

where the constant ω is the free vibration frequency of the beam.

The solution of Eqs (3.3) reads

$$T(t) = C_1 \sin \omega t + C_2 \cos \omega t$$

Taking into account Eqs (3.2), Eqs (3.1) are transformed to the form

$$\begin{aligned} &< B > W_0^{IV}(x) + < Bh'' > Q_0''(x) - \omega^2 < \rho > W_0(x) = 0 \\ &< Bh'' > W_0''(x) + \Big[< Bh''h'' > -\omega^2 < \rho hh >\Big] Q_0(x) = 0 \end{aligned} \tag{3.4}$$

After defining

$$B^{\rm eff} \equiv < B > - \frac{< Bh'' >^2}{< Bh''h'' >}$$

and introducing into Eqs (3.4) the micro-vibration frequency μ we obtain

$$\begin{aligned} &W_0^{IV}(x) - k^4 W_0(x) = 0 \\ &Q_0(x) = -\frac{< Bh'' >}{< Bh''h'' > \Big[1 - \left(\frac{\omega}{\mu}\right)^2\Big]} W_0''(x) \end{aligned} \tag{3.5}$$

where the eigenvalue k is defined as

$$k^4 = \frac{\frac{<\rho>}{<B>}\omega^2 \Big[1 - \left(\frac{\omega}{\mu}\right)^2\Big]}{\frac{B^{\rm eff}}{<B>} - \left(\frac{\omega}{\mu}\right)^2} \tag{3.6}$$

The expression (3.6) can be written also in the form

$$\omega^2 = \frac{B^{\text{eff}}}{<\rho>}k^4 + \Big[\omega^2 - \frac{<B>}{<\rho>}k^4\Big]\Big(\frac{\omega}{\mu}\Big)^2 \tag{3.7}$$

The second term of Eq (3.7) describes the dispersion effect due to the micro-periodicity of a beam. For a homogeneous beam with the constant cross-section $B^{\text{eff}} =< B >= B$ Eq (3.7) yields $\omega^2 = \frac{B}{\rho}k^4$. If $l \to 0$ than $\mu \to \infty$ and the dispersion effect disappears.

The solutions of Eqs (3.5) are

$$\begin{aligned} W_0(x) &= D_1 \sin kx + D_2 \cos kx + D_3 \sinh kx + D_4 \cosh kx \\ Q_0(x) &= \frac{< Bh'' > k^2}{< Bh''h'' > \Big[1 - \Big(\frac{\omega}{\mu}\Big)^2\Big]}(D_1 \sin kx + D_2 \cos kx - \\ &- D_3 \sinh kx - D_4 \cosh kx) \end{aligned} \tag{3.8}$$

Let the function $W_0(x)$ satisfy the boundary conditions for a simply supported beam

$$\begin{aligned} W_0(0) &= 0 \qquad & W_0''(0) &= 0 \\ W_0(L) &= 0 \qquad & W_0''(L) &= 0 \end{aligned} \tag{3.9}$$

In this case solution (3.8) reduces to

$$\begin{aligned} W_0(x) &= D_1 \sin \frac{m\pi x}{L} \\ Q_0(x) &= \frac{\frac{2<Bh''>}{<Bh''h''>}\Big(\frac{m\pi x}{L}\Big)^2}{1 - \Big(\frac{\omega_m}{\mu}\Big)^2 \mp \sqrt{\Big[\Big(\frac{\omega_m}{\mu}\Big)^2 + 1\Big]^2 - 4\Big(\frac{\omega_m}{\mu}\Big)^2 \frac{B^{\text{eff}}}{<B>}}} D_1 \sin \frac{m\pi x}{L} \end{aligned} \tag{3.10}$$

where $m = 1, 2, \ldots$ and D_1 is an arbitrary constant and

$$\omega_m = \Big(\frac{m\pi}{L}\Big)^2 \sqrt{\frac{< B >}{<\rho>}}$$

4. Steady state periodic vibrations

We consider steady state periodic vibrations of a beam. We assume loading of a beam in the form

$$f(x,t) = e^{i\widetilde{\omega}t}F(x) \tag{4.1}$$

where $\tilde{\omega}$ is the frequency of a loading process.

The response of the beam can be assumed in the form

$$
\begin{aligned}
W(x,t) &= \widetilde{W}(x)e^{i\widetilde{\omega}t} \\
Q^b(x,t) &= \widetilde{Q}^b(x)e^{i\widetilde{\omega}t}
\end{aligned}
\tag{4.2}
$$

Taking into account Eqs (4.1) and (4.2), Eqs (2.13) yield

$$
\begin{aligned}
&< B > \widetilde{W}^{IV}(x) + < Bh_b'' > \widetilde{Q}^{b\prime\prime}(x) - < \rho > \tilde{\omega}^2 \widetilde{W}(x) = < F > (x) \\
&< Bh_a'' > \widetilde{W}''(x) + < Bh_a''h_b'' > \widetilde{Q}^b(x) - < \rho h_a h_b > \tilde{\omega}^2 \widetilde{Q}^b(x) = < Fh_a > (x)
\end{aligned}
\tag{4.3}
$$

where a, b take the values $1, 2, ..., n$.

Let us consider a simply supported beam. The solution of the system of equations (4.3) can be represented by

$$
\begin{aligned}
\widetilde{W}(x) &= \sum_{m=1}^{\infty} W_m \sin \frac{m\pi x}{L} \\
\widetilde{Q}^b(x) &= \sum_{m=1}^{\infty} Q_m^b \sin \frac{m\pi x}{L}
\end{aligned}
\tag{4.4}
$$

Series (4.4) have to satisfy the boundary conditions

$$
\begin{aligned}
\widetilde{W}(0) &= 0 \qquad & \widetilde{W}''(0) &= 0 \\
\widetilde{W}(L) &= 0 \qquad & \widetilde{W}''(L) &= 0
\end{aligned}
\tag{4.5}
$$

Substituting RHS of (4.4) into Eqs (4.3) we obtain (after the simple calculations) the following system of linear algebraic equations

$$
\begin{aligned}
&\Big[< B > \Big(\frac{m\pi}{L}\Big)^4 - < \rho > \tilde{\omega}^2\Big] W_m - < Bh_b'' > \Big(\frac{m\pi}{L}\Big)^2 Q_m^b = \frac{2}{L} \ll F \gg_m \\
&- < Bh_a'' > \Big(\frac{m\pi}{L}\Big)^2 W_m + \Big[< Bh_a''h_b'' > - < \rho h_a h_b > \tilde{\omega}^2\Big] Q_m^b = \frac{2}{L} \ll Fh_a \gg_m
\end{aligned}
\tag{4.6}
$$

where

$$
\ll F \gg_m \equiv \int\limits_0^L < F > \sin \frac{m\pi x}{L}\, dx
$$

$$
\ll Fh_a \gg_m \equiv \int\limits_0^L < Fh_a > \sin \frac{m\pi x}{L}\, dx
$$

Introducing only one micro-shape function $h(x) = h_1(x)$, Eqs (4.6) reduce to

$$\begin{aligned}
&\Big[<B> \Big(\frac{m\pi}{L}\Big)^4 - <\rho> \widetilde{\omega}^2\Big] W_m - <Bh''> \Big(\frac{m\pi}{L}\Big)^2 Q_m = \frac{2}{L} \ll F \gg_m \\
&- <Bh''> \Big(\frac{m\pi}{L}\Big)^2 W_m + \Big[<Bh''h''> - <\rho hh> \widetilde{\omega}^2\Big] Q_m = \frac{2}{L} \ll Fh \gg_m
\end{aligned} \tag{4.7}$$

The solutions of Eqs (4.7) are

$$\begin{aligned}
W_m &= \frac{2\Big[\Big(1-\frac{\widetilde{\omega}^2}{\mu^2}\Big) \ll F \gg_m + \frac{<Bh''>}{<Bh''h''>}\Big(\frac{m\pi}{L}\Big)^2 \ll Fh \gg_m\Big]}{L <\rho> \omega_m^2\Big[\Big(1-\frac{\widetilde{\omega}^2}{\omega_m^2}\Big)\Big(1-\frac{\widetilde{\omega}^2}{\mu^2}\Big) - \Big(1-\frac{B^{\rm eff}}{<B>}\Big)\Big]} \\
Q_m &= \frac{2\Big[<Bh''> \Big(\frac{m\pi}{L}\Big)^2 \ll F \gg_m + <\rho> \omega_m^2\Big(1-\frac{\widetilde{\omega}^2}{\omega_m^2}\Big) \ll Fh \gg_m\Big]}{L <\rho> \omega_m^2 <Bh''h''> \Big[\Big(1-\frac{\widetilde{\omega}^2}{\omega_m^2}\Big)\Big(1-\frac{\widetilde{\omega}^2}{\mu^2}\Big) - \Big(1-\frac{B^{\rm eff}}{<B>}\Big)\Big]}
\end{aligned} \tag{4.8}$$

where

$$\omega_m = \Big(\frac{m\pi}{L}\Big)^2 \sqrt{\frac{<B>}{<\rho>}}$$

Hence the resonance frequency equals

$$\omega_{m_1}, \omega_{m_2} = \frac{\mu}{\sqrt{2}} \sqrt{\Big(\frac{\omega_m}{\mu}\Big)^2 + 1 \pm \sqrt{\Big[\Big(\frac{\omega_m}{\mu}\Big)^2 + 1\Big]^2 - 4\Big(\frac{\omega_m}{\mu}\Big)^2 \frac{B^{\rm eff}}{<B>}}} \tag{4.9}$$

For an uniformely distributed load $F(x) = F =$ const we have

$$\begin{aligned}
&<F> = F \qquad && \ll F \gg_m = \frac{FL}{m\pi}[1-(-1)^m] \\
&<Fh> = 0 \qquad && \ll Fh \gg_m = 0
\end{aligned}$$

Taking into account only the first term of both series (4.4) we recive

$$\begin{aligned}
\widetilde{W}(x) &= \frac{4F\Big(1-\frac{\widetilde{\omega}^2}{\mu^2}\Big)}{\pi <\rho> \omega_1^2\Big[\Big(1-\frac{\widetilde{\omega}^2}{\omega_1^2}\Big)\Big(1-\frac{\widetilde{\omega}^2}{\mu^2}\Big) - \Big(1-\frac{B^{\rm eff}}{<B>}\Big)\Big]} \sin\frac{\pi x}{L} \\
\widetilde{Q}(x) &= \frac{4F <Bh''> \pi}{L^2 <\rho> \omega_1^2\Big[\Big(1-\frac{\widetilde{\omega}^2}{\omega_1^2}\Big)\Big(1-\frac{\widetilde{\omega}^2}{\mu^2}\Big) - \Big(1-\frac{B^{\rm eff}}{<B>}\Big)\Big]} \sin\frac{\pi x}{L}
\end{aligned} \tag{4.10}$$

An alternative solution of Eqs (4.3) can be presented in the closed form. Assuming as an example $F(x) = F =$ const and taking into account the boundary conditions (4.5) we obtain solution of Eqs (4.3) in the following closed form

$$
\begin{aligned}
\widetilde{W}(x) &= \Big\{\Big[[(\sin\widetilde{k}L + \sinh\widetilde{k}L)(1-\cosh\widetilde{k}L)(\cos\widetilde{k}L - \cosh\widetilde{k}L) - \\
&- (\cos\widetilde{k}L - \cosh\widetilde{k}L)^2](\sin\widetilde{k}x - \sinh\widetilde{k}x) + \\
&+ \Big[(1-\cosh\widetilde{k}L)[(\cos\widetilde{k}L - \cosh\widetilde{k}L)^2 + (\sin^2\widetilde{k}L - \sinh^2\widetilde{k}L)] - \\
&- (\sin\widetilde{k}L - \sinh\widetilde{k}L)[(\sin\widetilde{k}L + \sinh\widetilde{k}L)(1-\cos\widetilde{k}L) - \\
&- (\cos\widetilde{k}L - \cosh\widetilde{k}L)]\Big](\cos\widetilde{k}x - \cosh\widetilde{k}x)\Big][2(1-\cos\widetilde{k}L\cosh\widetilde{k}L)\cdot \\
&\cdot (\cos\widetilde{k}L - \cosh\widetilde{k}L)]^{-1} - (1-\cosh\widetilde{k}x\Big\}\frac{F}{<\rho>\widetilde{\omega}^2}
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
\widetilde{Q}(x) &= \frac{<Bh''>}{<Bh''h''>}\frac{k^2}{\Big(1-\frac{\widetilde{\omega}^2}{\mu^2}\Big)}\frac{F}{<\rho>\widetilde{\omega}^2}\cdot \\
&\cdot \Big\{\Big[[(\sin\widetilde{k}L + \sinh\widetilde{k}L)(1-\cosh\widetilde{k}L)(\cos\widetilde{k}L - \cosh\widetilde{k}L) - \\
&- (\cos\widetilde{k}L - \cosh\widetilde{k}L)^2](\sin\widetilde{k}x + \sinh\widetilde{k}x) + \\
&+ \Big[(1-\cosh\widetilde{k}L)[(\cos\widetilde{k}L - \cosh\widetilde{k}L)^2 + (\sin^2\widetilde{k}L - \sinh^2\widetilde{k}L)] - \\
&- (\sin\widetilde{k}L - \sinh\widetilde{k}L)[(\sin\widetilde{k}L + \sinh\widetilde{k}L)(1-\cos\widetilde{k}L) - \\
&- (\cos\widetilde{k}L - \cosh\widetilde{k}L)]\Big](\cos\widetilde{k}x + \cosh\widetilde{k}x)\Big][2(1-\cos\widetilde{k}L\cosh\widetilde{k}L)\cdot \\
&\cdot (\cos\widetilde{k}L - \cosh\widetilde{k}L)]^{-1} - \cosh\widetilde{k}x\Big\}
\end{aligned}
$$

where

$$
\widetilde{k}^4 = \frac{\frac{<\rho>}{<B>}\widetilde{\omega}^2\Big[1-\Big(\frac{\widetilde{\omega}}{\mu}\Big)^2\Big]}{\frac{B^{\rm eff}}{<B>} - \Big(\frac{\widetilde{\omega}}{\mu}\Big)^2}
\tag{4.12}
$$

5. Final remarks

From the analysis presented above it follows that the proposed macro-dynamics of elastic beam structures can be effectively applied to engineering problems. This line of approach can be applied to more complicated dynamic problems as well. Some of these problems will be investigated in subsequent papers.

References

1. BAKHVALOV N.S., PANASENKO G.P., 1984, *Osrednenie processov v perëdicheskikh sredakh,* (in Rusian), Nauka, Moskva

2. WOŹNIAK C., 1993, *Micro-macro dynamics of periodic material structures,* 2nd European Conf. on Struct. Dynamics, Eurodyn'93, Trondheim, Norway

3. WOŹNIAK. C., *Refined macro-dynamics of periodic structures,* Arch.Mech.Stos., in press

Makro-dynamika mikro-periodycznych belek sprężystych

Streszczenie

Celem pracy jest przedstawienie pewnej metody makro-modelowania mikroperiodycznych sprężystych belek. Sposób podejścia do problemu jest oparty na nieasymptotycznej metodzie makro-modelowania zaproponowanej przez Woźniaka [3].

W niniejszej pracy otrzymuje się równania ruchu dla prostej, liniowo sprężystej belki o strukturze mikroperiodycznej.

Przedstawia się rozwiązanie dynamicznego zagadnienia własnego i drgań ustalonych belki w ramach inżynierskiej teorii zginania belek sprężystych.

Manuscript received May 12, 1993; accepted for print May 17, 1993