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ON NON-LINEAR ANALYSIS OF THE GEARED DRIVE SYSTEMS BY MEANS OF THE WAVE METHOD

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In the paper non-linear torsional vibrations of the geared drive systems are investigated. There are considered effects of backlashes in the gear stages, variations of the tooth stiffnesses, wear and manufacture errors of teeth as well as effects of various models of damping application. Considerations are performed using a discrete-continuous mechanical model. An application of the d'Alembert solutions of the wave motion equations leads to an appropriate system of linear and non-linear ordinary differential equations with a "shifted" argument. Numerical integration of these equations is very efficient, stable and accurate.

1. Introduction

The vibration problems associated with gear stages of the drive systems of machines and vehicles have been difficult to deal with from an engineering viewpoint. An operation of the gear stages is usually affected by torsional, lateral, longitudinal and circumferential vibrations of the drive system elements. Backlashes in the gear stages, variation of the tooth stiffnesses, wear and manufacture errors as well as a complex character of damping have a significant influence on vibrations of the whole drive system. These vibrations result in high stresses in shafts, couplings and gear teeth and in a generation of unnecessary noise. As it follows from (Laschet, 1988; Neryia et al., 1987; Kubo, 1987; Chou and Yang, 1987; Pfeiffer and Kunert, 1990), for investigations of gear dynamics torsional vibrations are the most important and couplings with remaining kinds of vibrations often can be neglected.

Computer simulations of the geared drive systems for an analysis of the vibration effects are mostly based on a discrete mechanical model described by coupled non-linear ordinary differential equations. But a numerical integration of these equations is usually time consuming (Laschet, 1988; Pfeiffer and Kunert, 1990;

Bogacz et al., 1992b) and often can bring significant errors, particularly in a case of impact investigations of the gear teeth due to backlashes (Chou and Yang, 1987; Pfeiffer and Kunert, 1990). Thus, in the present paper there is proposed an alternative method of non-linear vibration analysis based on a discrete-continuous mechanical model and on the wave interpretation of the vibration phenomenon.

2. Assumptions and formulation of the problem

A subject of considerations in the paper is the drive system of a machine driven by the electric motor or the internal combustion reciprocating engine by means of n single-stage gears. It is assumed that torsional vibrations in the system are predominant and an influence of other kinds of vibrations is negligible. For the torsional vibration analysis a discrete-continuous mechanical model is employed. It consists of $2n+2$ rigid bodies of constant or variable mass moments of inertia $I_1, I_j^{(m)}, I_{n+2}, j = 2,3,...,n+1, m = 1,2$, respectively, $n+1$ torsionally deformable cylindrical elastic elements with continuously distributed parameters of lengths l_i and constant torsional stiffnesses k_i, $i = 1,2,...,n+1$, as well as of n massless torsional springs of variable stiffnesses h_j, $j = 2,3,...,n+1$, Fig.1. The rigid bodies represent mass moments of inertia of rotating parts of the driving motor, $2n$ gear wheels and of the driven machine rotor, respectively. The elastic elements with distributed parameters correspond to the shaft segments. While the massless torsional springs represent flexibility of the gear teeth. The considered system is excited to vibrations by the imposed on rigid body (1) active external torque $M_1(t)$ produced by the driving motor as well as by the imposed on rigid body $(n+2)$ passive external torque $M_{n+2}(t)$ received by the driven machine. Absolute damping and shaft material damping in the system are represented by a linear model of the viscous type in the form of equivalent damping moments imposed on the rigid bodies. Moreover, for the gear stages, additional non-linear damping terms with variable coefficients e_j are introduced.

Commonly applied drive systems are usually equipped with elastic couplings, joints, friction clutches and others. A more detailed description of the discrete-continuous model of the drive system with the aforementioned elements one can find in Bogacz et al. (1992b).

An essential problem is to select an appropriate model representing stiffness and damping properties of the gear stages, which reduces to the assumption of proper functions for h_j and e_j. In the classical backlash model of the gear stage $h_j(\Delta\Theta_j) = h_{0j} = \text{const}$ for $|\Delta\Theta_j| > \varepsilon$ and $h_j(\Delta\Theta_j) = 0$ for $|\Delta\Theta_j| \le \varepsilon$ (Laschet, 1988; Neryia et al., 1987), where ε is a constant value of the tooth backlash angle and $\Delta\Theta_j$ is a difference between angles of rotation of the rigid body $j-(1)$ and $j-(2)$, $j = 2,3,...,n+1$, Fig.1. If the variation of tooth stiffness is considered,

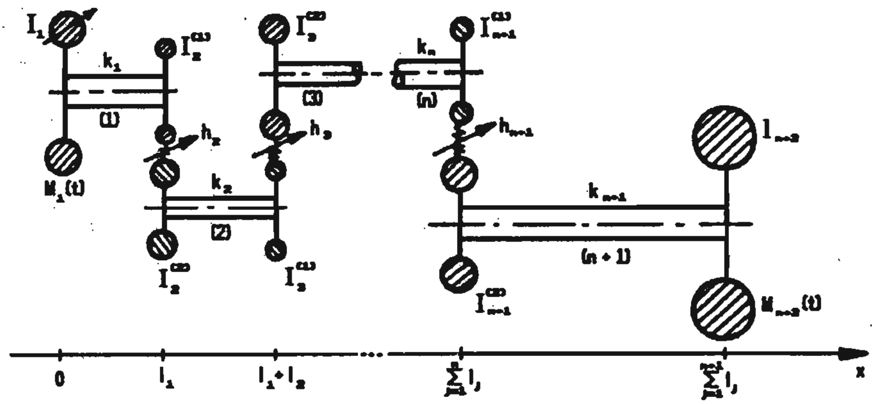

Fig. 1. Discrete-continuous model of the geared drive system

$h_j(\Delta\Theta_j)$ can be expressed by sinusoidal or trapezoidal functions (cf Laschet, 1988; Neryia et al., 1987; Kubo, 1987). In practice due to wear and manufacture errors of teeth ε is not a constant value. Because of difficulties in establishing real backlash values, ε is usually assumed as a sum of the harmonic deterministic component and the random component expressed by the filtered white noise function generated in the form of solution of the following equation (Neryia et al., 1987)

$$\ddot{\varepsilon}_r(t) + b_1\dot{\varepsilon}_r(t) + b_2\varepsilon_r(t) = G(t)$$

where $G(t)$ is the Gauss white noise process, b_1, b_2 are appropriate filter coefficients and t denotes time. The coefficients of damping in the gear stages e_j are usually expressed as parabolic or exponential functions of $\Delta\Theta_j$ (Laschet, 1988).

Currently, to the gear dynamics analysing also various impact theories are applied. For example, Chou and Yang (1987) employ the Hertzian contact theory and thus the stiffnesses h_j and damping coefficients e_j are expressed as

$$h_j(\Delta\Theta_j) = \pi E W_j \frac{m(\Delta\Theta_j)}{4(1-\nu^2)}$$

$$e_j(\Delta\Theta_j, \Delta\dot{\Theta}_j) = 6(1-e_0)\frac{h_j(\Delta\Theta_j)}{(2(e_0-1)^2+3)\Delta\dot{\Theta}_j}$$

where
E – Young modulus
W_j – product of the tooth thickness and square of the jth driven wheel radius, $j = 2, 3, ..., n+1$
m – number of tooth pairs in contact
ν – Poisson ratio
e_0 – coefficient of restitution.

Equations of motion for angular displacements of the elastic element cross-sections are classical wave equations

$$a^2\Theta_{i,xx}(x,t) - \Theta_{i,tt}(x,t) = 0 \qquad i = 1, 2, ..., n+1 \tag{2.1}$$

where: $a^2 = G/\rho$ and x is a spatial coordinate parallel to the system rotation axis, Fig.1. These equations are solved with the assumed initial conditions (e.g. homogeneous) and with the following boundary conditions

$$I_1(\Theta_1)\Theta_{1,tt} + \Big[d_1 + \frac{1}{2}\Theta_{1,t}L_1(\Theta_1)\Big]\Theta_{1,t} - c_1 l_1 \Theta_{1,xt} - k_1 l_1 \Theta_{1,x} = M_1(t) \qquad \text{for} \quad x = 0$$

$$\begin{aligned}
&I_j^{(1)}\Theta_{j-1,tt} + d_j^{(1)}\Theta_{j-1,t} + c_{j-1}l_{j-1}\Theta_{j-1,xt} + \kappa_j e_j(\Delta\Theta_j)[\kappa_j\Theta_{j-1,t} - \Theta_{j,t}] + \\
&+k_{j-1}l_{j-1}\Theta_{j-1,x} + \kappa_j h_j(\Delta\Theta_j)[\kappa_j\Theta_{j-1} - \Theta_j] = M_j^{(1)} \\
&I_j^{(2)}\Theta_{j,tt} + d_j^{(2)}\Theta_{j,t} - c_j l_j \Theta_{j,xt} - e_j(\Delta\Theta_j(t))[\kappa_j\Theta_{j-1,t} - \Theta_{j,t}] - k_j l_j \Theta_{j,x} - \\
&-h_j(\Delta\Theta_j(t))[\kappa_j\Theta_{j-1} - \Theta_j] = M_j^{(2)} \\
&\Delta\Theta_j(t) = \kappa_j\Theta_{j-1} - \Theta_j \qquad j = 2, 3, ..., n+1 \qquad \text{for} \quad x = \sum_{i=1}^{j-1} l_i
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&I_{n+2}\Theta_{n+1,tt} + d_{n+2}\Theta_{n+1,t} + c_{n+1}l_{n+1}\Theta_{n+1,xt} + k_{n+1}l_{n+1}\Theta_{n+1,x} = \\
&= M_{n+2}(t) \qquad \text{for} \quad x = \sum_{i=1}^{n+1} l_i
\end{aligned}$$

where

$$L_1(\Theta_1) = \frac{dI_1(\Theta_1)}{d\Theta_1} \qquad M_j^{(1)} = -\mathrm{sgn}(\Delta\Theta_j)\varepsilon_j \frac{h_j(\Delta\Theta_j)}{\kappa_j}$$

$$M_j^{(2)} = \mathrm{sgn}(\Delta\Theta_j)\varepsilon_j h_j(\Delta\Theta_j)$$

κ_j – gear ratios, $j = 2, 3, ..., n+1$
d_k – constant absolute damping coefficient, $k = 1, 2, ..., n+2$
c_l – constant shaft material damping coefficient, $l = 1, 2, ..., n+1$.

Superscripts (1) and (2) are assigned to the quantities corresponding to the driving and driven elements in the system, respectively, Fig.1. The subscripts after commas denote partial differentiations.

Solutions of Eqs (2.1) are sought for in the form of the d'Alembert solutions

$$\Theta_i(x,t) = f_i(at - x + \sum_{j=1}^{i-1} l_j) + g_i(at + x - \sum_{j=1}^{i-1} l_j) \qquad i = 1,2,...,n+1 \quad (2.3)$$

The functions f_i and g_i in Eq (2.3) represent torsional waves propagating in the elastic elements as a result of the external torque application. They are determined by the boundary and the initial conditions (Bogacz et al., 1992a,b). Thus, substitution of Eq (2.3) into the boundary conditions (2.2) leads to the following system of linear and non-linear ordinary differential equations with a "shifted" argument z for the functions f_i and g_i, $i = 1,2,...,n+1$

$$r_{2,n+2}g''_{n+1}(z) + r_{1,n+2}g'_{n+1}(z) = M_{n+2}(z - l_{n+1}) + s_{2,n+2}f''_{n+1}(z - 2l_{n+1}) + \\ + s_{1,n+2}f'_{n+1}(z - 2l_{n+1})$$

$$r_{21}(z)f''_1(z) + r_{11}(z)f'_1(z) = M_1(z) + s_{21}(z)g''_1(z) + s_{11}(z)g'_1(z) \qquad (2.4)$$

$$\begin{bmatrix} p_{2,j1} & 0 \\ 0 & r_{2,j} \end{bmatrix} \begin{bmatrix} g''_{j1}(z + l_{j1}) \\ f''_j(z) \end{bmatrix} + \begin{bmatrix} p_{1,j1}(z) & -\kappa_j e_j(\Delta_j(z)) \\ -\kappa_j e_j(\Delta_j(z)) & r_{1,j}(z) \end{bmatrix} \cdot \\ \cdot \begin{bmatrix} g'_{j1}(z + l_{j1}) \\ f'_j(z) \end{bmatrix} + \begin{bmatrix} \kappa_j^2 h_j(\Delta_j(z)) & -\kappa_j h_j(\Delta_j(z)) \\ -\kappa_j h_j(\Delta_j(z)) & h_j(\Delta_j(z)) \end{bmatrix} \begin{bmatrix} g_{j1}(z + l_{j1}) \\ f_j(z) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where

$$B_1 = M_j^{(1)} + u_{2,j1}f''_{j1}(z - l_{j1}) + u_{1,j1}(z)f'_{j1}(z - l_{j1}) + \kappa_j e_j(\Delta_j(z))g'_j(z) + \\ + \kappa_j h_j(\Delta_j(z))[g_j(z) - \kappa_j f_{j1}(z - l_{j1})]$$

$$B_2 = M_j^{(2)} + s_{2,j}g''_j(z) + s_{1,j}g'_j(z) + \kappa_j e_j(\Delta_j(z))f'_{j1}(z - l_{j1}) - \\ - h_j(\Delta_j(z))[g_j(z) - \kappa_j f_{j1}(z - l_{j1})]$$

$$\Delta_j(z) = \kappa_j[f_{j-1}(z - l_{j-1}) + g_{j-1}(z + l_{j-1})] - f_j(z) - g_j(z)$$

$$j = 2,3,...,n+1 \qquad\qquad j1 = j - 1$$

$$r_{2,n+2} = c_{n+1}l_{n+1} + aI_{n+2} \qquad r_{1,n+2} = \frac{l_s(k_{n+1}l_{n+1} + ad_{n+2})}{a}$$

$$s_{2,n+2} = c_{n+1}l_{n+1} - aI_{n+2} \qquad s_{1,n+2} = \frac{l_s(k_{n+1}l_{n+1} - ad_{n+2})}{a}$$

$$r_{21}(z) = c_1l_1 + aI_1(z) \qquad r_{11}(z) = \frac{l_s[k_1l_1 + a(d_1 + \Omega_1(z)L_1(z))]}{a}$$

$$s_{21}(z) = c_1 l_1 - a I_1(z) \qquad s_{11}(z) = \frac{l_s[k_1 l_1 - a(d_1 + \Omega_1(z) L_1(z))]}{a}$$

$$p_{2,j1} = c_{j1} l_{j1} + a I_j^{(1)} \qquad p_{1,j1}(z) = \frac{l_s[k_{j1} l_{j1} + a(d_j^{(1)} + \kappa_j^2 e_j(\Delta_j(z)))]}{a}$$

$$u_{2,j1} = c_{j1} l_{j1} - a I_j^{(1)} \qquad u_{1,j1}(z) = \frac{l_s[k_{j1} l_{j1} - a(d_j^{(1)} - \kappa_j^2 e_j(\Delta_j(z)))]}{a}$$

$$r_{2,j} = c_j l_j + a I_j^{(2)} \qquad r_{1,j}(z) = \frac{l_s[k_j l_j + a(d_j^{(2)} + e_j(\Delta_j(z)))]}{a}$$

$$s_{2,j} = c_j l_j - a I_j^{(2)} \qquad s_{1,j}(z) = \frac{l_s[k_j l_j - a(d_j^{(2)} - e_j(\Delta_j(z)))]}{a}$$

$$\Omega_1(z) = \frac{1}{2}[f_1'(z) + g_1'(z)] \qquad j1 = j - 1 \qquad j = 2, 3, ..., n+1$$

and l_s is an arbitrary value. Using the Newmark method to solve Eq (2.4) together with Eq (2.3) one obtains a system transient or steady-state dynamic response in the form of torques or tangential stresses in the shafts and in the form of forces or angular accelerations of the gear teeth. The "shifted" argument in Eq (2.4), which is a consequence of the wave interpretation of torsional vibrations, makes their right hand sides always known at each computation step. Thus, in contrary to coupled differential equations for an analogous discrete mechanical model, it is possible to solve Eq (2.4) sequentially, one after another, in the presented order. As it follows from (Bogacz et al., 1992a,b), this feature very essentially simplifies the numerical procedure making it much more efficient, stable and accurate.

3. Numerical example

In the numerical example there was performed a start-up simulation of a machine driven by the asynchronous motor by means of $n = 4$ gear stages. The system was accelerated from a rest to its nominal rotational speed. In this example for simplicity and clarity of results only variable tooth stiffness in the gear stages is considered. The external torque $M_1(t)$ produced by the motor was assumed according to (Laschet, 1988). Time history of this torque is showed in Fig.2. However, the load torque $M_6(t)$ of the driven machine was assumed in the following form: $M_6(t) = -(2.8 \cdot 10^5)t$ [Nm] for $t \leq 0.5$ [s] and $M_6(t) = -1.4 \cdot 10^5 = \text{const}$ [Nm] for $t > 0.5$ [s]. The gear stiffnesses $h_j(\Delta\Theta_j)$, $j = 2, 3, 4, 5$, are described by harmonic functions (cf Laschet, 1988; Kubo, 1987).

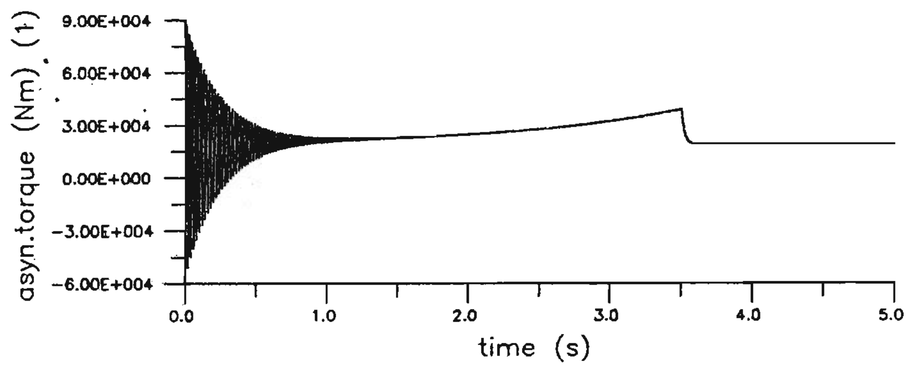

Fig. 2. Torque produced by the asynchronous motor

However, for the damping coefficients $e_j(\Delta\Theta_j)$ parabolic functions were assumed (cf Laschet, 1988).

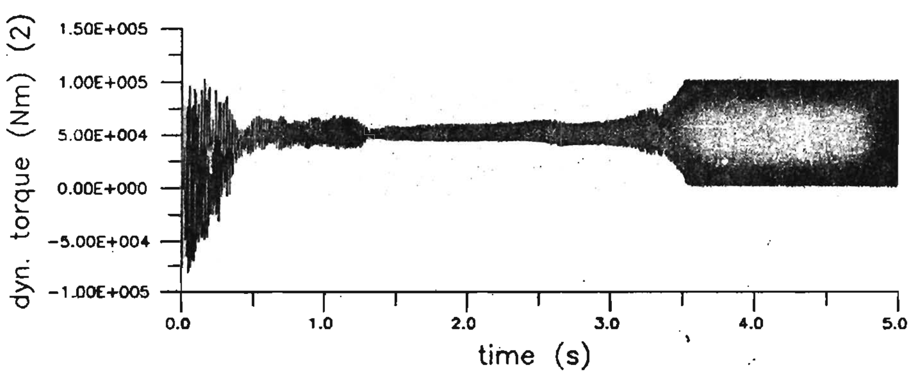

Fig. 3. Dynamic torque transmitted through the first gear (2) – in time domain

Fig.3 and 4 present system response in the form of dynamic torque transmitted through the first gear stage (2) in time and frequency domain, respectively. However, Fig.5 shows a history of the dynamic torque transmitted through the second gear stage (3). From Fig.3 and 4 it follows, that after 3.5 seconds of the start-up, when the nominal rotational speed was achieved, the system fell into resonance excited by the first gear stage stiffness variation of the frequency equal to 618 [Hz], which is close to the fourth natural frequency with the node location just in the first gear stage (2). However, in the second gear stage, together with the aforementioned resonance, an influence of the additional excitation due to variation of the stiffness h_3 is visible, Fig.5.

The amplitude spectrum in Fig.4 shows, that for the first gear stage an influence

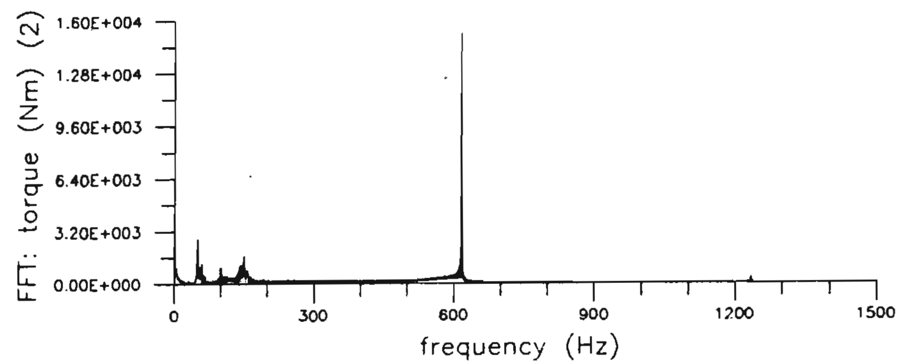

Fig. 4. Dynamic torque transmitted through the first gear (2) – in frequency domain

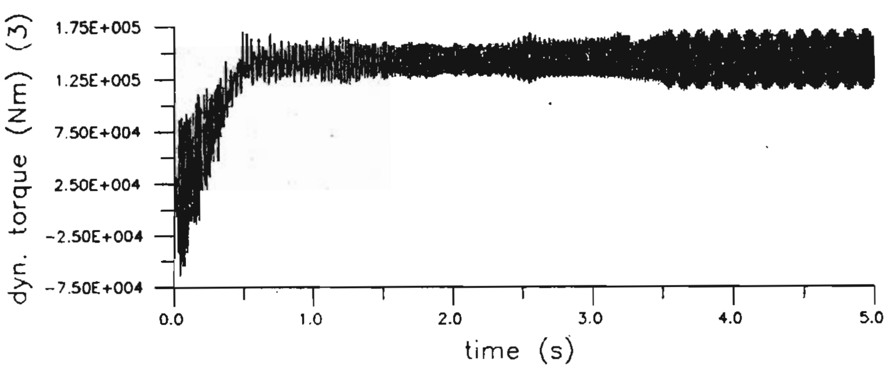

Fig. 5. Dynamic torque transmitted through the second gear (3)

of the pulsating component of the asynchronous motor torque of frequency 50 [Hz] is relatively small, but vibrations excited by a variable tooth stiffness due to the gear meshing are predominant. In this example vibrations caused by the variable tooth stiffnesses are also predominant in the remaining gear stages and they can be particularly dangerous for fatigue process of the tooth material.

4. Final remarks

In the presented paper a discrete-continuous mechanical model and the torsional wave propagation theory were applied to investigation of dynamic effects in the geared drive systems. Using the presented procedure a consideration of backlash effects, tooth stiffness variations, wear and manufacture errors as well as of various damping models reduces to a selection of an appropriate gear stiffness and damping coefficient functions for a numerical simulation. In the proposed method all the aforementioned properties of the gear stages are included in the boundary conditions of the problem. Because of an application of the d'Alembert method, the boundary conditions determine motion of the system. The obtained in consequence ordinary differential equations with a "shifted" argument are easy for the efficient numerical integration and achieving high computational stability and accuracy.

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O analizie nieliniowej przekładniowych układów napędowych przy pomocy metody falowej

Streszczenie

W pracy badane są nieliniowe drgania skrętne przekładniowych układów napędowych. Rozpatruje się efekty w przekładniach zębatych wywoływane luzami, zmiennością sztywności zazębienia, zużyciem i błędami wykonawczymi, jak również efekty wywoływane przyjmowaniem różnych modeli tłumienia. Badania te są przeprowadzane za pomocą dyskretno-ciągłego modelu mechanicznego. Wykorzystanie rozwiązań d'Alemberta falowych równań ruchu prowadzi do odpowiedniego układu liniowych i nieliniowych równań różniczkowych zwyczajnych z "przesuniętym" argumentem. Całkowanie tych równań jest bardzo efektywne numerycznie, stabilne i dokładne.

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