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I STOSOWANA
Journal of Theoretical
and Applied Mechanics
2, 31, 1993

ON THE CERTAIN ANALYTICAL EXAMPLE OF THE POTENTIAL FLOW IN A SEMI-INFINITE PERIODIC DOMAIN

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The analytic solution to the Neumann boundary problem for the Laplace equation in a space-periodic domain has been obtained. This domain is the exterior of the y-periodic palisade of contours from the 2-parametric family. Such solution can beused as a test for the accuracy checking of numerical algorithms.

1. Formulation of the problem in physical and transformed domain

The general task is as follows

- Find the y-periodic solution to the Laplace equation in the domain Ω presented in Fig.1, satisfying the Neumann conditions on the boundary $\partial\Omega$. In terms of the ideal fluid dynamics it is the problem of determining the velocity potential having the normal velocity on the given boundary. Additionally one requires that the flow far a way from the front of the palisade is a homogeneous stream with prescribed velocity V_∞.

Generally, two different approaches can be applied here

1. solving the problem directly in a physical domain; the main difficulty is to assure the y-periodicity of the solution and the fact that domain is infinitely multi-connected is also an obstacle,

2. transforming the domain, using some conformal mapping, so that y-periodicity would be obtained automatically.

The first approach has been discussed by Szumbarski (1993). Hereinafter the second one is useful, with the obvious observation that the mapping we need is the simply exponent function (according to Fig.1. the y-period is equal to 2π). Then every contour of the palisade is mapped onto the same closed curve and the

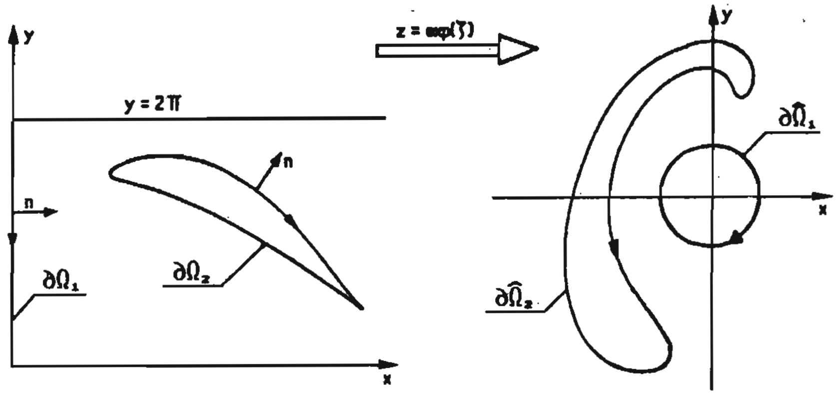

Fig. 1. The exponential conformal mapping transforms y-periodic domain to double-connected, unbounded region

inlet line $(x = 0)$ – onto the unit circle. Thus the problem has been reduced to solving the Laplace equation in a double-connected, unbounded region $\widehat{\Omega}$ with transformed Neumann conditions on the boundary $\partial\widehat{\Omega}$ (Fig.1).

In general case, in order to find a solution, some numerical methods should be employed. The analytical treatment is possible when, for instance, we assume that

1. the contours of the palisade $(\partial\Omega_2)$ are mapped onto a circle,
2. there is no inlet line i.e. the physical domain Ω is extended (to the left) to infinity. Thus $\partial\Omega_1$ is reduced to the origin.

If in the domain Ω a certain velocity at infinity (in front of the palisade) is prescribed $\boldsymbol{V}_\infty = [u_\infty, v_\infty]$ then, after transformation, one obtains at the origin

– the source has the flux

$$Q = 2\pi u_\infty \tag{1.1}$$

– the vortex circulation of which will be determined below on.

Thus the flow past the cylinder due to the presence of the singularity of the origin has been obtained in the transformed domain $\widehat{\Omega}$. Its complex potential can be constructed with the use of the Milne-Thomson theorem (cf Milne-Thomson, 1952)

- If $W(z)$ is a certain complex potential, the circle of radius R with a center at the origin is the one of the flow stream lines given by potential of the form

$$W_c(z) = W(z) + W\Big(\frac{R^2}{z}\Big)$$

Two cases are of particular interest[1]

a) $W(z) = \frac{Q}{2\pi} \ln z$ – source at the origin

b) $W(z) = \frac{\Gamma}{2\pi i} \ln z$ – vortex at the origin

If we consider the conFiguration shown in Fig.2a, it can be calculated (see Appendix A) that

$$W_C(z) = \frac{Q}{2\pi} \ln(z-b) + \frac{Q}{2\pi} \ln\left(z - \frac{R^2}{b}\right) - \frac{Q}{2\pi} \ln z \tag{1.2}$$

Hence, $W_C(z)$ is the superposition of three sources. Similarly in the case shown in Fig.2b, the following formula can be calculated

$$W_c(z) = \frac{\Gamma}{2\pi i} \ln(z-b) + \frac{\Gamma}{2\pi i} \ln z - \frac{\Gamma}{2\pi i} \ln\left(z - \frac{R^2}{b}\right) \tag{1.3}$$

Hence, $W_C(z)$ is the superposition of three vortices.

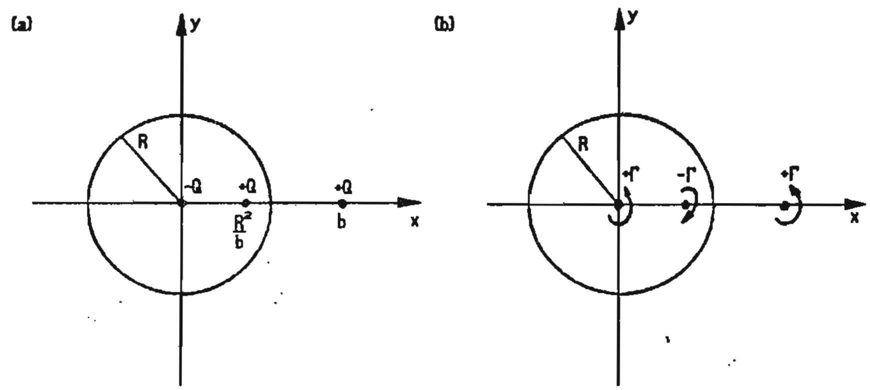

Fig. 2. The configuration of the singularities of the flow in the exterior of the circle induced by: (a) – single source, (b) – single vortex

However, unlike Eq (1.2), the formula (1.3) is only one of many possible choices. It is easy to see that the second term in Eq (1.3) represents the vortex located in the center of the circle. Thus no normal velocity is induced by this vortex regardless of its circulation.

We consider the following 2-parametric family of circles

$$z(\alpha) = -d + re^{-i(\alpha-\pi)} \tag{1.4}$$

[1]For certain reason it is more convenient to consider these cases separately instead of introducing the single value $Q - i\Gamma$

The complex coordinates of singular points can be introduced

$$\begin{array}{lcl} z_0 = 0 & - & \text{origin, omitted below} \\ z_1 = -d & - & \text{center of the circle} \\ z_2 = -d + \frac{r^2}{d} & - & \text{inverse of the origin with} \\ & & \text{respect to the circle center} \end{array} \tag{1.5}$$

Then, applying Eq (1.3), one obtains

$$W_C(z) = \frac{Q}{2\pi}\Big\{\ln z - \ln(z - z_1) + \ln(z - z_2)\Big\} \tag{1.6a}$$

for the source flow, and

$$W_C(z) = \frac{\Gamma}{2\pi \mathrm{i}}\Big\{\ln z + \ln(z - z_1) - \ln(z - z_2)\Big\} \tag{1.6b}$$

for the vortical flow.

2. The flow in the physical domain

Application of the inverse transformation $\Big(ln(z)\Big)$ to Eq (1.4) generates the 2-parametric family of oval contours in the physical domain. Since this transformation is multivalued we obtain the y-periodic palisade. Some examples of contours from this family are shown in Fig.3.

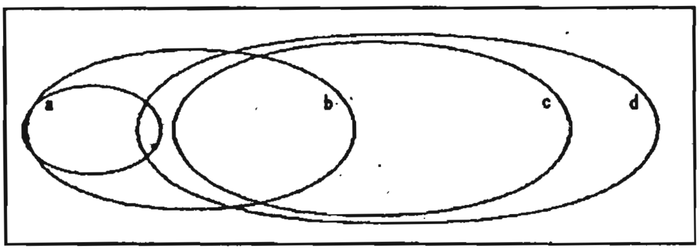

Fig. 3. Four examples of the contours for various choices of the parameters r and d: a – $r = 2.0$, $d = 3.2$; b – $r = 20.0$, $d = 21.5$; c – $r = 200.0$, $d = 206$; d – $r = 500.0$, $d = 504.0$

The following formula defines the complex velocity field in the physical domain Ω

$$v(\zeta) = zV(z) \tag{2.1}$$

where $z = \exp(\zeta)$, $\zeta \in \Omega$. The formula (2.1) is implied by the general formula

$$v(\zeta) = \frac{V(z)}{F(z)} \qquad \zeta = F(z)$$

where $F(z) = \ln z$ in this case.

The velocity field $V(z)$ in the auxiliary domain $\widehat{\Omega}$ is obtained by derivation of Eq (1.6)

$$V_Q(z) = \frac{Q}{2\pi}\Big[\frac{1}{z} - \frac{1}{z-z_1} + \frac{1}{z-z_2}\Big] \tag{2.2a}$$

– source-flow velocity

$$V_\Gamma(z) = \frac{\Gamma}{2\pi \mathrm{i}}\Big[\frac{1}{z} + \frac{1}{z-z_1} - \frac{1}{z-z_2}\Big] \tag{2.2b}$$

– vortex-flow velocity.

To find the flow in Ω the circulation Γ should be determined in the way ensuring that the condition imposed on the velocity at infinity is satisfied. It can be shown (cf Chmielniak, 1989) that an infinite y-periodic "palisade" of identical vortices (called y-periodic vortex from now on) induces the velocity field due to the formula

$$V_\Gamma(\zeta) = \frac{\Gamma}{4\pi \mathrm{i}} \coth\frac{\zeta-\zeta_0}{2} \tag{2.3}$$

where ζ, $\zeta_0 \in \Omega$, ζ_0 – complex location of the vortex with $0 \le \mathrm{Im}\zeta_0 \le 2\pi$. This velocity does not vanish at infinity

$$\lim_{\mathrm{Re}(\zeta)\to-\infty} V_\Gamma(\zeta) = -\frac{\Gamma}{4\pi \mathrm{i}} \qquad\qquad \lim_{\mathrm{Re}(\zeta)\to+\infty} V_\Gamma(\zeta) = \frac{\Gamma}{4\pi \mathrm{i}}$$

Thus, at left-side-infinity the vertical velocity equals $-\frac{\Gamma}{4\pi}$, and at right-side-infinity it has the same absolute value but an opposite sign. Hence we have

$$\Gamma = -4\pi V_\infty \tag{2.4}$$

Moreover the equivalent vortical flow in $\widehat{\Omega}$ should be taken in the modified form

$$V_\Gamma(z) = \frac{\Gamma}{4\pi \mathrm{i}}\Big[-\frac{1}{z} + \frac{1}{z-z_1} + \frac{1}{z-z_2}\Big] \tag{2.5}$$

(which is also geometrically admissible) due to the fact, that the y-periodic vortex flow is transformed by an exponent function into the flow induced by the pair of vortices: the first with Γ circulation at the center of circle, and the second one with $-\frac{\Gamma}{2}$ circulation at the origin (see Appendix B).

Finally, the total velocity field in the physical domain Ω is calculated by superposition

$$v(\zeta) = v_Q(\zeta) + v_\Gamma(\zeta) \tag{2.6}$$

where v_Q, v_Γ, Q and Γ are given by Eqs (2.2a), (2.5), (2.4), (1.1), respectively. Two examples of flow that can be obtained are shown in Fig.4 and 5.

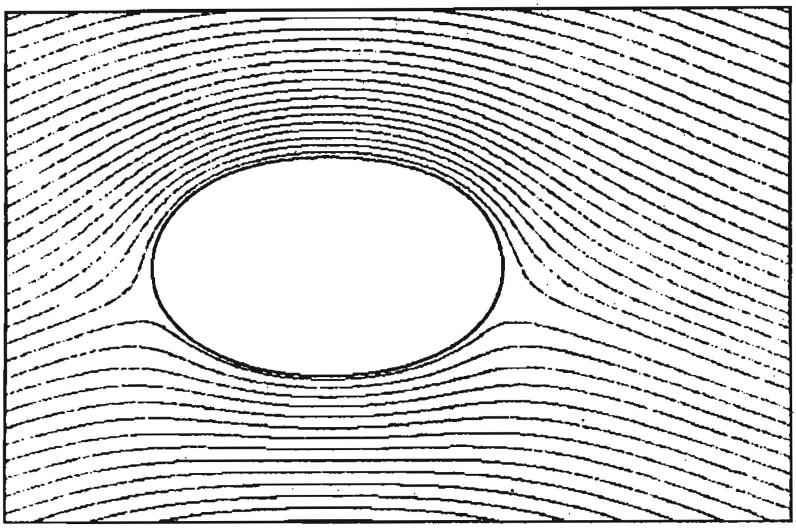

Fig. 4. The example of the flow in the palisade of contours for $r = 200$ and $d = 206$. The velocity at infinity is (1.0,0.2)

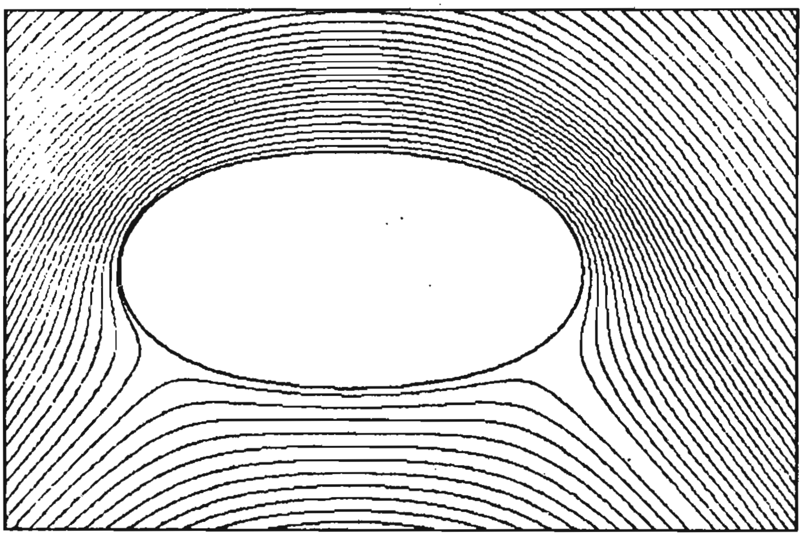

Fig. 5. The example of the flow in the palisade of contours for $r = 500$ and $d = 504$. The velocity at infinity is (1.0,0.7)

3. Conclusion

It has been shown that the potential of the infinite, y-periodic palisade of the airfoils belonging to certain 2-parametric family can be obtained analytically in the closed form. The Milne-Thomson theorem has been used with a special care in choosing the proper form of vortical component of the flow field. The obtained family of flows can be treated as a convenient testing example for numerical methods designed to solve the problem formulated in the beginning of the present paper (cf Szumbarski, 1993). Also the case of flows within semi-infinite domains with the vertical inlet line can be considered – one needs only to calculate analytically the normal velocity distribution over the inlet and employ it as a boundary condition in numerical algorithms.

Appendix

A The derivation of formulas (1.2) and (1.3)

We show that the formulas (1.2) and (1.3) hold. To prove Eq (1.2) we start with the complex potential function for the source

$$W(z) = \frac{Q}{2\pi}\ln(z-b) \tag{A.1}$$

Thus the flux of the source is Q and its location is $(b,0)$. Accordingly to the Milne-Thomson theorem

$$W_C(z) = W(z) + \overline{W\Big(\frac{R^2}{\overline{z}}\Big)}$$

one obtains

$$\begin{aligned} W_C(z) &= \frac{Q}{2\pi}\ln(z-b) + \frac{Q}{2\pi}\overline{\ln\Big(\frac{R^2}{\overline{z}}-b\Big)} = \\ &= \frac{Q}{2\pi}\ln(z-b) + \frac{Q}{2\pi}\ln\Big(\frac{R^2}{z}-b\Big) \end{aligned} \tag{A.2}$$

where the obvious identity $\overline{\ln z} = \ln\overline{z}$ has been used.

Since the velocity is determined by the differentiation of W_C, an arbitrary constant can be added to (A.2). We chose this constant to be equal to $\frac{Q}{2\pi}\ln\Big(-\frac{1}{b}\Big)$.

Then

$$
\begin{aligned}
W_C(z) &= \frac{Q}{2\pi}\ln(z-b)+\frac{Q}{2\pi}\ln\Big(\frac{R^2}{z}-b\Big)+\frac{Q}{2\pi}\ln\Big(-\frac{1}{b}\Big)= \\
&= \frac{Q}{2\pi}\ln(z-b)+\frac{Q}{2\pi}\ln\Big[\Big(\frac{R^2}{z}-b\Big)\Big(-\frac{1}{b}\Big)\Big]= \\
&= \frac{Q}{2\pi}\ln(z-b)+\frac{Q}{2\pi}\ln\Big(z-\frac{R^2}{b}\Big)-\frac{Q}{2\pi}\ln z
\end{aligned}
\tag{A.3}
$$

(since $\left(b-\frac{R^2}{z}\right)\frac{1}{b}=\left(z-\frac{R^2}{b}\right)/z$ in accordance with Eq (1.2). Similarly the formula (1.3) can be derived. After application of the Milne-Thomson theorem the constant equal to $\frac{\Gamma}{2\pi i}\ln(-\frac{1}{b})$ should be subtract and after simple calculations Eq (1.3) is obtained.

B Comments on the formula (2.5)

Accordingly to Eq (2.1) we have

$$
V(z)=\frac{v(\zeta)}{z}
$$

where $\zeta=\ln z$. Thus

$$
\begin{aligned}
V_\Gamma(z) &= \frac{\Gamma}{4\pi i z}\coth\frac{\zeta(z)-\zeta_0(z)}{2}=\frac{\Gamma}{4\pi i z}\frac{e^{\zeta(z)-\zeta_0(z)}+1}{e^{\zeta(z)-\zeta_0(z)}-1}= \\
&= \frac{\Gamma}{4\pi i z}\frac{\frac{z}{z_0}+1}{\frac{z}{z_0}-1}=\frac{\Gamma}{4\pi i}\frac{z+z_0}{z(z-z_0)}=\frac{\Gamma}{2\pi i}\frac{1}{z-z_0}-\frac{\Gamma/2}{2\pi i}\frac{1}{z}
\end{aligned}
$$

i.e. the flow in the physical domain in transformed into the flow induced by the pair of vortices in the auxiliary domain (Z). In order to eliminate the normal velocity induced on the circle, the third vortex with the circulation $\Gamma/2$ has to be added to $z_1=-d+r^2/d$. Moreover the circulation of the vortex located at z_0 should be taken as $\Gamma/2$ rather than Γ, which allows to obtain the proper value of the y-component of the velocity in the physical domain, at infinity behind the contours of the palisade.

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Analiza przepływu potencjalnego w obszarze okresowym

Streszczenie

W artykule omówiono przykład analitycznego rozwiązania zagadnienia wyznaczania potencjalnego pola prędkości w stopniu okresowej palisady konturów z pewnej dwuparametrycznej rodziny. Przy konstrukcji rozwiązania użyto odwzorowania konforemnego $z = \exp(W)$ a następnie twierdzenia Milne-Thomsona. Otrzymane rozwiązanie może służyć jako test dokładności metod przybliżonych wyznaczania przeplywów potencjalnych w obszarach o geometrii okresowej.

Manuscript received December 1, 1992; accepted for print December 31, 1992