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RIGID SLIDING PUNCH ON A PERIODIC TWO-LAYERED ELASTIC HALF-SPACE

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The paper deals with the plane problem of a sliding punch with friction along the surface of a microperiodic two-layered elastic half-space. The analysis is performed within the framework of the homogenized model with microlocal parameters.

1. Introduction

This paper is a sequel to our earlier study (cf Kaczyński and Matysiak, 1988a) concerning the plane static contact problems with friction for a microperiodic two-layered elastic composite. These investigations referred to pertain to symmetric indentations made by a single smooth punch with the given region of contact beforehand (so-called complete contact, cf Gladwell, 1980). The present work aims at the generalization about some problems of incomplete penetrations, where one or both ends of the contact region are unknown and have to be determined by requiring that the normal stresses are bounded at these points. The analysis is carried out within the linear plane-strain static theory of elasticity with microlocal parameters developed by Woźniak (1987), Matysiak and Woźniak (1987) and being useful in solving several types of boundary value problems for periodic elastic composites (cf Kaczyński and Matysiak, 1988b,c and 1989). The obtained below results can be applied in the geotechnical engineering (foundations problems on periodic two-layered structures such as varved clays, flotation wastes, cf Kaczyński and Matysiak (1992) as well as sandstone-slate, sandstone-shale, thin-layered limestone) and certain problems of the tribology.

In Section 2 the problem of a sliding punch with friction along the surface of a microperiodic half-space is posed. On the basis of the results obtained by

Muskhelishvili (1953), Galin (1980), Goryacheva and Dobykhin (1988), Kaczyński and Matysiak (1988a) the distribution of normal stress under the punch is deduced in the case of general contact region and by taking the two-component limiting friction law.

Two particular examples by assuming different shapes of punch (with straight inclined and parabolic base) and regions of contact regarding the influence of the layering and friction are solved in Section 3.

2. Problem formulation

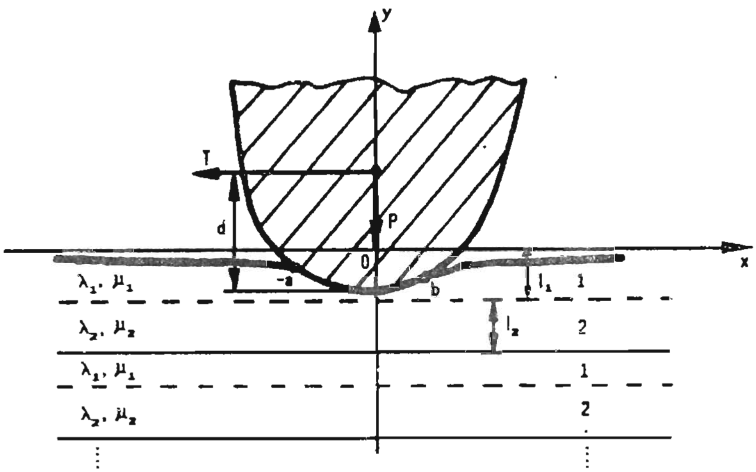

Fig. 1.

We consider the plane static contact problem of a two-layered microperiodic elastic half-space with a rigid smooth punch. The scheme of the middle cross-section is given in Fig.1. Let λ_1, μ_1 and λ_2, μ_2 be Lame constants of the subsequent layers (perfect bonding between them is assumed). To analyse the punch problem on the layered body we take into consideration the homogenized model of linear elasticity with microlocal parameters given by Woźniak (1987), Matysiak and Woźniak (1987). The governing equations of it are formulated in terms of the unknown macrodisplacement vector and certain extra unknowns being referred to as microlocal parameters. We will apply the notations and results presented in our previous study (cf Kaczyński and Matysiak, 1988a).

Similarly to the aforementioned paper (cf Kaczyński and Matysiak, 1988a) consider the plane-strain contact problem of a sliding rigid punch along the surface of two-layered periodically laminated half-space. However, we shall refer to the situation of asymmetrical indentation (see Fig.1) assuming that over the whole interval $(-a,b)$ the following two-component law of limiting friction holds (cf Goryacheva and Dobykhin, 1988)

$$\sigma_{xy}^{(1)}(x,0^-)=\tau_0-k\sigma_{yy}^{(1)}(x,0^-) \tag{2.1}$$

where $\sigma_{yy}^{(1)}$ and $\sigma_{xy}^{(1)}$ are normal and shear stresses, respectively, under the punch, k and τ_0 are constants, the first called the coefficient of dry friction and the second regarding the effect of adhesion in friction formation (the upper index 1 is related to the layers with material constants λ_1, μ_1; so $\sigma_{yy}^{(2)}$, $\sigma_{xy}^{(2)}$, $\sigma_{xx}^{(2)}$ denote the stresses in the layers with material constants λ_2, μ_2, respectively).

We examine the state of equilibrium of the punch produced by the action of given vertical force P and the horizontal force T

$$T\equiv\int_{-a}^{b}\sigma_{xy}^{(1)}(x,0^-)dx$$

applied at the distance d from a punch base.

Referring to Fig.1 for the location of the coordinate system connected with the punch, the boundary conditions of the considered problem take the form

$$\begin{aligned} &v(x,0^-)=f(x)+\text{const.} && \text{for}\quad x\in<-a,b> \\ &\sigma_{xy}^{(1)}(x,0^-)=\tau_0-k\sigma_{yy}^{(1)}(x,0^-) && \text{for}\quad x\in(-a,b) \\ &\sigma_{yy}^{(1)}(x,0^-)=\sigma_{xy}^{(1)}(x,0^-)=0 && \text{for}\quad x\in\mathcal{R}-<-a,b> \end{aligned} \tag{2.2}$$

$$\sigma_{\alpha\beta}^{(j)}(x,y)\to 0\quad\text{for}\quad\sqrt{x^2+y^2}\to\infty \qquad (\alpha,\beta=x\ \text{or}\ y;\quad j=1\ \text{or}\ 2)$$

and in addition

$$\begin{aligned} &\int_{-a}^{b}\sigma_{yy}^{(1)}(x,0^-)dx=-P && \text{with}\quad P>0 \\ &p(x)\equiv-\sigma_{yy}^{(1)}(x,0^-)>0 && \forall x\in(-a,b) \end{aligned} \tag{2.3}$$

In the first one of conditions (2.2) the function $f(\cdot)$ describing the normal displacement $v(x,0^-)$ is assumed to be known from the shape of the punch end-face and the type of indentation.

Following the complex variable approach presented by Muskhelishvili (1953), Galin (1980), Goryacheva and Dobykhin (1988), Kaczyński and Matysiak (1988a) it can be shown for the considered case of the contact problem described by boundary conditions (2.2) and (2.3) that the pressure $p(x)$, $x \in (-a,b)$, under the punch takes the form

$$
\begin{aligned}
p(x) &= -\sin(\pi\beta)\cos(\pi\beta)F(x) + \frac{1}{\pi}\cos(\pi\beta)\Big[P + \cos(\pi\beta)\cdot \\
&\cdot \int\limits_{-a}^{b} \frac{F(\tau)(\tau+a)^{\frac{1}{2}+\beta}(b-\tau)^{\frac{1}{2}-\beta}}{\tau - x} d\tau\Big](x+a)^{-\frac{1}{2}-\beta}(b-x)^{-\frac{1}{2}+\beta}
\end{aligned}
\tag{2.4}
$$

where

$$
F(x) = \frac{A\tau_0}{A_+} + Af'(x) \tag{2.5}
$$

Here the constants A, A_+ and β,related to the periodic structure of the intendent body, are given by Kaczyński and Matysiak (1988a) by the formulae

$$
\begin{aligned}
A &= A_+ \frac{\sqrt{A_- C}}{\sqrt{A_2(A_+ + 2C)}} \\
\beta &= \frac{1}{\pi}\arctan\Big(k\frac{A}{A_+}\Big) \in <0;\frac{1}{2})
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
A_\pm &= \sqrt{A_1 A_2} \pm B \\
A_1 &= \frac{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}{(1-\eta)(\lambda_1 + 2\mu_1) + \eta(\lambda_2 + 2\mu_2)} \\
A_2 &= A_1 + \frac{4\eta(1-\eta)(\mu_1 - \mu_2)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)}{(1-\eta)(\lambda_1 + 2\mu_1) + \eta(\lambda_2 + 2\mu_2)} \\
B &= \frac{(1-\eta)\lambda_2(\lambda_1 + 2\mu_1) + \eta\lambda_1(\lambda_2 + 2\mu_2)}{(1-\eta)(\lambda_1 + 2\mu_1) + \eta(\lambda_2 + 2\mu_2)} \\
C &= \frac{\mu_1\mu_2}{(1-\eta)\mu_1 + \eta\mu_2} \\
\eta &= \frac{l_1}{l_1 + l_2}
\end{aligned}
\tag{2.7}
$$

and l_1, l_2 are thicknesses of the subsequent layers (see Fig.1).

3. Analysis of two indentation cases

We shall examine thoroughly two cases of penetrations by assuming different shapes of punch and regions of contact, respectively. Similar analysis corresponding to a contact of a rigid punch with the homogeneous half-plane may be found elsewhere (cf Muskhelishvili, 1953; Galin, 1980; Goryacheva and Dobykhin, 1988).

3.1. Punch with straight inclined base

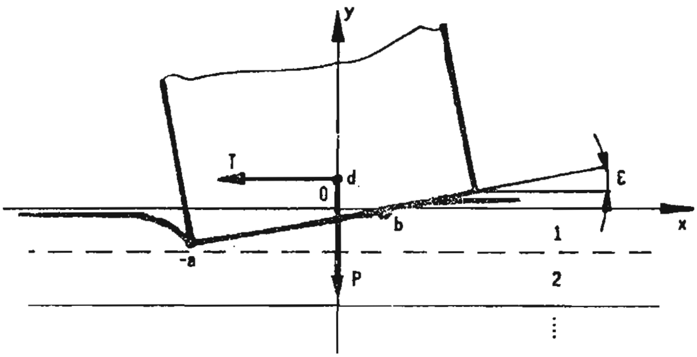

Fig. 2.

Consider the case of indentation of the microperiodic two-layered elastic half-space by a flat ended rigid punch under the action of normal force P and tangential force T applied in such a way that the base of the punch remains tilted at the angle ε as shown in Fig.2. Then we put in Eq $(2.2)_1$

$$f(x) = \varepsilon x \tag{3.1}$$

and from Eq (2.5) it appears that the function F becomes constant written as follows

$$F(x) = A\Big(\frac{T_0}{A_+} + \varepsilon\Big) \equiv m \tag{3.2}$$

From the general formula (2.4),after performing the integration (for details refer to Muskhelishvili, 1953; Galin, 1980; Goryacheva and Dobykhin, 1988), we find

$$p(x) = \frac{\cos\pi\beta}{\pi}\,\frac{P + \pi m\Big[b - x - (a+b)\Big(\frac{1}{2} + \beta\Big)\Big]}{(x+a)^{\frac{1}{2}+\beta}(b-x)^{\frac{1}{2}-\beta}} \tag{3.3}$$

This asymmetrical pressure distribution in the region of contact $(-a, b)$ leads to a couple of moment M about the origin, which is found to be given by

$$M \equiv \int_{-a}^{b} xp(x)dx = P(a+b)\Big(\frac{1}{2}-\beta\Big) - Pa - \frac{\pi m}{2}(a+b)^2\Big(\frac{1}{4}-\beta^2\Big) \tag{3.4}$$

In the equilibrium state of the punch under the action of the normal force P and the tangential force T

$$T \equiv \int_{-a}^{b} \sigma_{xy}^{(1)}(x,0^-)dx = \tau_0(a+b) + kP \tag{3.5}$$

applied at the distance d from the base (see Fig.2), the following condition of equality of the moments holds

$$M + Td = 0 \tag{3.6}$$

which in view of Eqs (3.4) and (3.5) may now be rewritten as

$$P(a+b)\Big(\frac{1}{2}-\beta\Big) - Pa - \pi\frac{m}{2}(a+b)^2\Big(\frac{1}{4}-\beta^2\Big) + \tau_0(a+b)d + kPd = 0 \tag{3.7}$$

From Eqs (3.7) and (3.2) the angle of tilt ε is found to be

$$\varepsilon = \frac{2\Big[P(a+b)\Big(\frac{1}{2}-\beta\Big) - Pa + \tau_0(a+b)d + kPd\Big]}{\pi A(a+b)^2\Big(\frac{1}{4}-\beta^2\Big)} - \frac{\tau_0}{A_+} \tag{3.8}$$

Consider first the case of complete indentation. Putting in Eqs (3.3) and (3.8) $b = a$ we have

$$p(x) = \frac{\cos(\pi\beta)}{\pi}\,\frac{P - \pi m(x+2a\beta)}{(x+a)^{\frac{1}{2}+\beta}(a-x)^{\frac{1}{2}-\beta}} \tag{3.9}$$

and

$$\varepsilon = \frac{(kP + 2a\tau_0)d - 2aP\beta}{2\pi Aa^2\Big(\frac{1}{4}-\beta^2\Big)} - \frac{\tau_0}{A_+} \tag{3.10}$$

The condition $(2.3)_1$ will be satisfied if

$$-\frac{\tau_0}{A_+} - \frac{P}{2\pi Aa\Big(\frac{1}{2}-\beta\Big)} \le \varepsilon \le \frac{P}{2\pi Aa\Big(\frac{1}{2}+\beta\Big)} - \frac{\tau_0}{A_+} \tag{3.11}$$

Combining Eq (3.11) with Eq (3.10) we obtain the condition for the distance d in order to realize the complete contact

$$0 \le d \le \frac{Pa\Big(\frac{1}{2}+\beta\Big)}{kP + 2a\tau_0} \equiv d_c \tag{3.12}$$

If this inequality holds, the angle of punch inclination ε is determined by Eq (3.10).

Note that

$$\begin{aligned}
&\varepsilon = 0 \Leftrightarrow d = \frac{2\left[aP\beta + \pi\frac{A}{A_+}a^2\tau_0\left(\frac{1}{4}-\beta^2\right)\right]}{kP + 2a\tau_0} \equiv d_0 \\
&d = 0 \Leftrightarrow \varepsilon = -\frac{P\beta}{\pi Aa\left(\frac{1}{4}-\beta^2\right)} - \frac{\tau_0}{A_+} \equiv \varepsilon_0 < 0 \\
&0 < d < d_0 \Leftrightarrow \varepsilon_0 < \varepsilon < 0 \\
&d_0 < d < d_c \Leftrightarrow 0 < \varepsilon < \frac{P}{2\pi Aa\left(\frac{1}{2}+\beta\right)} - \frac{\tau_0}{A_+} \equiv \varepsilon_c
\end{aligned} \tag{3.13}$$

We turn next to a case of incomplete penetration. To this end the following asymptotic behaviour of unbounded stresses at the edges of the contact region, as $r \to 0^+$, is examined to yield dominant terms

$$p(-a+r) = \frac{K_{-a}}{r^{\frac{1}{2}+\beta}} + 0(r^{\frac{1}{2}-\beta}) \tag{3.14}$$

$$p(b-r) = \frac{K_b}{r^{\frac{1}{2}-\beta}} + 0(r^{\frac{1}{2}+\beta}) \tag{3.15}$$

where

$$K_{-a} \equiv \frac{\cos(\pi\beta)}{\pi}\,\frac{P + m\pi(a+b)\left(\frac{1}{2}-\beta\right)}{(a+b)^{\frac{1}{2}-\beta}} \tag{3.16}$$

$$K_b \equiv \frac{\cos(\pi\beta)}{\pi}\,\frac{P - m\pi(a+b)\left(\frac{1}{2}+\beta\right)}{(a+b)^{\frac{1}{2}+\beta}} \tag{3.17}$$

From Eq (3.17) it follows that for the case of $b = a$ and $d = d_c$ corresponding to $\varepsilon = \varepsilon_c$ we have $K_a = 0$ and $p(a) = 0$. Hence for $d > d_c$ a state of incomplete contact will result. In this case we can apply Eqs (3.10) and (3.17) to calculate the point of detachment of the punch from the intended foundation

$$b = -a + \frac{2P(a-kd)}{P\left(\frac{1}{2}-\beta\right) + 2\tau_0 d} \qquad b \in \langle -a, a) \tag{3.18}$$

which substituted into Eq (3.8) leads to

$$\varepsilon = \frac{P\left(\frac{1}{2}-\beta\right) + 2\tau_0 d}{2\pi A\left(\frac{1}{2}+\beta\right)(a-kd)} - \frac{\tau_0}{A_+} \tag{3.19}$$

It is seen that for $d = a/k \equiv d_u$ we have $b = -a$, $\varepsilon \to \infty$, what correspond to the case when the punch is tilted over.

It may also be confirmed by equations (3.9) and (3.14) that $p(x) > 0$ for every $d \in < 0, d_u)$ and $x \in (-a, b)$ provided that $|b| < a$. Besides, if $d = d_c$ formulae (3.10) and (3.19) for ε yield the same results.

The analysis performed is illustrated by a graph of the angle of tilt ε versus the distance $d \in < 0, d_u)$ in Fig.3.

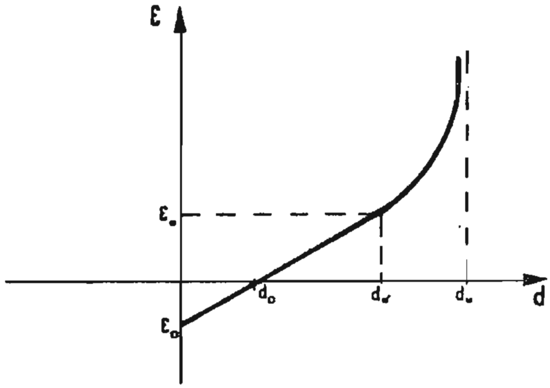

Fig. 3.

It can be noticed that in the absence of friction $(k = 0,\ \beta = 0,\ \tau_0 = 0)$ the above obtained results reduce to that given by Muskhelishvili (1953), Galin (1980), Goryacheva and Dobykhin (1988).

3.2. Punch with parabolic base

As a second case we consider the problem of incomplete indentation by assuming in Eq $(2.2)_1$ that

$$f(x) = \frac{x^2}{2R} \qquad\qquad R = \text{const} > 0 \tag{3.20}$$

which corresponds to the sliding of a rigid parabolic punch (or it can be used to the approximation of a circular punch with large radius R). Referring to Eqs (2.5) and (3.20) we obtain

$$F(x) = A\Big(\frac{\tau_0}{A_+} + \frac{x}{R}\Big) \qquad\qquad \text{for} \qquad x \in (-a, b) \tag{3.21}$$

Substituting Eq (3.21) into Eq (2.4) and performing integration (the details may be found elsewhere, cf Muskhelishvili, 1953; Galin, 1980; Goryacheva and Dobykhin, 1988) we have

$$p(x) = \cos(\pi\beta)\frac{H(x)}{(x+a)^{\frac{1}{2}+\beta}(b-x)^{\frac{1}{2}-\beta}} \tag{3.22}$$

where

$$\begin{aligned} H(x) &= \frac{P}{\pi} + \frac{A(a+b)^2\left(\frac{1}{4}-\beta^2\right)}{2R} + \frac{A\tau_0(a+b)\left(\frac{1}{2}-\beta\right)}{A_+} + \\ &+ \frac{A\left(\frac{1}{2}-\beta\right)(a+b)x}{R} - \frac{A\tau_0(x+a)}{A_+} + \frac{Ax(x+a)}{R} \end{aligned} \tag{3.23}$$

At this stage we confine our attention to the dominant asymptotic behaviour of the pressure at the ends $-a$ and b. As $r \to 0^+$, we obtain

$$p(-a+r) = \cos(\pi\beta)\frac{H(-a)+O(r)}{(a+b-r)^{\frac{1}{2}-\beta}\, r^{\frac{1}{2}+\beta}} \tag{3.24}$$

$$p(b-r) = \cos(\pi\beta)\frac{H(b)+O(r)}{(a+b-r)^{\frac{1}{2}+\beta}\, r^{\frac{1}{2}-\beta}} \tag{3.25}$$

The condition that the normal stresses are bounded at the edges of the contact region leads to the following set of equations

$$H(-a) = 0 \qquad\qquad H(b) = 0 \tag{3.26}$$

which, after using the denotation for the length of the contact by $l \equiv a+b$, may be rewritten as

$$l^2 = \frac{2RP}{\pi A\left(\frac{1}{4}-\beta^2\right)} \tag{3.27}$$

$$\frac{a-b}{2} = l\beta + \frac{R\tau_0}{A_+} \tag{3.28}$$

From the above formulae it is easily seen that the adhesion component of friction τ_0 plays a part only in the increase of the displacement $\frac{1}{2}(a-b)$ of the contact region in relation to the symmetry axis of the sliding punch and does not affect the extent of the contact length l.

Corresponding to the values a and b, determined from the solution of the set of equations (3.27) and (3.28), we find that the formula (3.23) takes now a simple form

$$p(x) = \frac{A}{R}\cos(\pi\beta)(x+a)^{\frac{1}{2}-\beta}(b-x)^{\frac{1}{2}+\beta} > 0 \qquad \forall\, x \in (-a,b) \tag{3.29}$$

Hence the conclusion that only the coefficient of friction k through β has an influence on the distribution of contact normal stresses.

Again as in the previous case we can calculate the moment M produced by the pressure $p(x)$

$$M \equiv \int\limits_{-a}^{b} xp(x)dx = -P\Big(4l\frac{\beta}{3} + \frac{\tau_0}{A_+}\Big) \tag{3.30}$$

which should be balanced against the moment of the moving force

$$T \equiv \int\limits_{-a}^{b} \sigma_{xy}(x,0^-)dx = \tau_0 l + kP \tag{3.31}$$

It yields the condition for the distance d defining the point of action of the force T (see Fig.1)

$$d = \Big|\frac{M}{T}\Big| \tag{3.32}$$

where M and T are given by Eqs (3.30) and (3.31), respectively.

Having analyzed the results obtained here, one can see that in the cases of practical interest $k(A/A_+) \ll 1$ and therefore (see Eq $(2.6)_2$) $\beta \approx kA/(\pi A_+) \ll 1$. Thus, the calculations performed without regard to friction do not lead to large errors.

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Sztywny ślizgający się stempel na periodycznej dwuwarstwowej półprzestrzeni

Streszczenie

W pracy rozpatrzono płaskie zagadnienie ślizgającego się stempla z uwzględnieniem tarcia po powierzchni mikroperiodycznej dwuwarstwowej półprzestrzeni. Analizę przeprowadzono w ramach homogenizowanego modelu z parametrami mikrolokalnymi.

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