MECHANIKA
TEORETYCZNA
I STOSOWANA
1/2, 22 (1984)

PLASTIC ZONE SIZE OF DUGDALE TYPE CRACKS IN A SELF-STRESSED TWO-PHASE MEDIUM WITH PARTIALLY PLASTIFIED MATRIX MATERIAL

KLAUS P. HERRMANN and IVAN M. MIHOVSKY [1]

Institute of Mechanics
University of Paderborn
4790 Paderborn, Federal Republic of Germany

Abstract

Different boundary-value problems are considered concerning the elastic-plastic and fracture behaviour of brittle fibres-ductile matrix composites under thermal loading conditions in both the cases of absence and presence of cracks within the matrix phase. A model of the plastic deformation process is proposed with regard to a single unit cell of the fibre-reinforced composite. Numerous details of the deformation process within this unit cell are investigated by use of the above mentioned model including the possible failure mechanisms of the fibre-matrix bond. If applied together with the known crack model of the Dugdale type the proposed model for the plastic deformation process of a composite unit cell is shown to imply useful conclusions concerning the thermal crack growth of radial Dugdale type cracks within the matrix phase.

Introduction

The investigation of the interaction between the stress fields caused by the presence of different inhomogeneities is a problem of great practical importance. This is actually the basic problem of the mechanics of the composite materials. Of special interest from the point of view of fracture mechanics of the composite structures are the questions concerning the interaction between the structural components and existing cracks within these structures. Both the cases of mechanical and thermal loading of cracked composites have been since long studied and different models of interaction have been already considered by means of both micromechanical analysis and macromechanical theories. The essential features of these two different approaches were characterized in a paper by SMITH [1]. The fibre-reinforced composites consisting of ductile matrices strengthened by continuous brittle fibres form a large class of the commonly used composite materials.

[1] On leave from *the Department of Mathematics and Mechanics, Sofia University, Sofia 1090, Bulgaria*

Thereby numerous investigations concerning the plastic behaviour of fibre-reinforced composites have been performed for example in the papers of HILL [2], SPENCER [3], MULHERN et al. [4], COOPER and PIGGOTT [5]. Comprehensive surveys about the state of the art are given in the Conference Proceedings of the 1975 ASME Winter Annual Meeting [6] as well as in the books of SPENCER [7], KOPIOV and OVCINSKIJ [8] and PIGGOTT [9]. Further, a problem of basic interest represents the micromechanical aspect in thermal cracking of unidirectionally reinforced composites. Thereby, definite progress has been already made in a series of papers by HERRMANN [10 - 12] and HERRMANN and associates [13 - 15] concerning the elastic and viscoelastic behaviour of a cracked unit cell of a low fibre concentration composite under the conditions of different thermal loading. In a recent work by HERRMANN and MIHOVSKY [16] the plastic behaviour of an uncracked unit cell and the mechanisms of failure of the fibre-matrix interface have been analyzed for the case of isothermal longitudinal extension of the composite. The model of the plastic deformation process proposed in [16] is especially attractive for the study of the behaviour of cracks situated within the matrix phase. It is shown in the present paper that this model is applicable to the problem of thermal loading of the composite. Moreover, if combined with the Dugdale model solution of HERRMANN [11] for a crack situated within the matrix phase this above mentioned model of the plastic deformation process implies useful conclusions concerning the fracture behaviour of the considered unit cell of a unidirectionally reinforced composite.

Statement of the problem

A unidirectionally reinforced fibrous composite with continuous fibres and relatively small fibre volume fraction is considered. The fibre material is linear elastic with Young's modulus E_f, Poisson's ratio ν_f and the thermal expansion coefficient α_f. The material of the matrix is elastic-perfectly plastic with corresponding elastic constants E_m and ν_m, thermal expansion coefficient α_m and tensile yield stress σ_y. The thermoelastic properties of the fibre and matrix materials as well as the yield stress of the latter are assumed to be temperature independent.

A unit cell of this fibre-reinforced composite in the sense of the well-known model of two coaxial fibre-matrix cylinders is studied in the following where if referred to a cylindrical coordinate system (r, θ, z) the fibre and the matrix occupy the regions $(0 \leqslant r \leqslant r_f,\ 0 \leqslant \theta \leqslant 2\pi,\ -\infty < z < +\infty)$ and $(r_f \leqslant r \leqslant r_m,\ 0 \leqslant \theta \leqslant 2\pi,\ -\infty < z < +\infty)$, respectively. Thus, equation $r = r_f$ is the equation of the fibre-matrix interface.

The following is assumed with regard to the cracked composite unit cell. The crack is situated within the matrix phase and is presented in the cross section of the unit cell (cf. Fig. 1) by a straight line cut along the polar axis $\theta = 0$. The crack occupies the segment $r_l \leqslant r \leqslant r_r$ so that r_l and r_r denote the radial coordinates of the left and the right crack tip, respectively. The crack length of the actual crack is $2l = r_r - r_l$.

The well-known elastic-plastic model of a crack proposed by DUGDALE [17] will be applied in the following analysis. The crack of length $2L = R_r - R_l$ shown in Fig. 1 presents the imaginary crack in the sense of this model. The segments $R_l \leqslant r \leqslant r_l$ and $r_r \leqslant r \leqslant R_r$

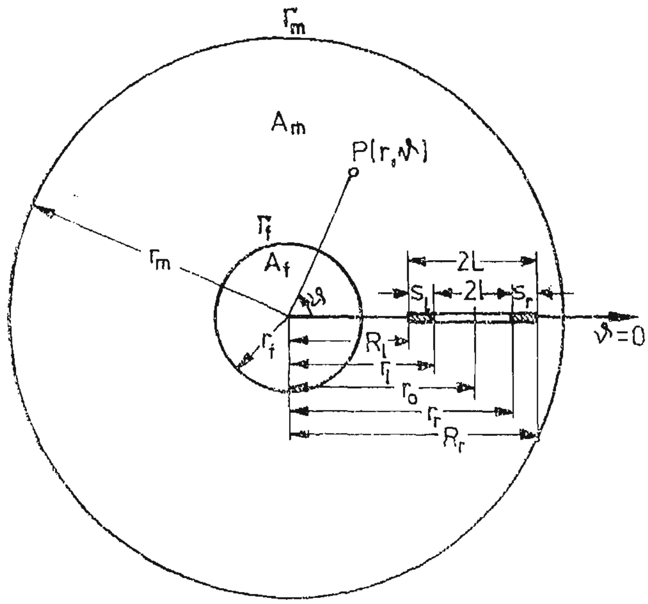

Fig. 1. Dugdale type crack configuration in the cross section of a composite unit cell

present the thin plastic zones at the tips of the actual crack. The plastic zone lengths at the left and the right crack tips are thus $s_l = r_l - R_l$ and $s_r = R_r - r_r$, respectively.

A quasi-static thermal loading of the unit cell will be considered which implies a temperature distribution over the cross section of the form (cf. Fig. 1 for notation)

$$T(r,\theta) = \begin{cases} T_f, & 0 \leqslant r \leqslant r_f \\ T_m, & r_f \leqslant r \leqslant r_m \end{cases} \tag{1}$$

The temperatures T_f and T_m do not depend on the axial coordinate z and are constants at each given instant of the deformation process so that the latter is viewed as a sequence of stress-strain states of the cell corresponding to a sequence of stationary temperature distributions of the form (1).

From the viewpoint of a quasi-static crack propagation behaviour such loading conditions are of interest only which result in the appearance of tensile circumferential stresses within the uncracked matrix phase. According to HERRMANN [18] these conditions are satisfied for the elastic state of the unit cell provided the inequality

$$(1+\nu_f)\alpha_f(T_f - T_0) - (1+\nu_m)\alpha_m(T_m - T_0) > 0, \tag{2}$$

holds true where T_0 is the temperature of the unstressed initial state. Under the simplifying assumption $T_0 = 0$ relation (2) is obviously satisfied if, for example, $T_f = 0$ and $T_m < 0$. This case will be actually considered in the following calculations. The thermal loading process will be thus viewed as a process of monotonous quasi-static decrase of the itself negative temperature of the matrix phase. The accepted loading conditions provide obviously an axisymmetric state of stress within the uncracked composite cell. The axial symmetry together with the standard assumptions of perfect fibre-matrix contact and generalized plane strain imply the evident result that the normal stresses within the uncracked unit cell are at the same time principal ones and depend on the radial coordinate only.

The elastic state of the considered unit cell for both cases of absence and presence of a crack in the matrix phase has been described in detail by Herrmann [11, 18]. These elastic solutions concern a cell with a traction-free external surface $r = r_m$ and traction-free or partially loaded crack surfaces in the sense of the applied Dugdale model. These same conditions are supposed to apply in the here considered elastic-plastic problem as well.

The condition of axial symmetry together with the assumed scheme of loading implies certain obvious features of the elastic-plastic state of the uncracked unit cell in accordance with the above mentioned elastic solution [11]. These are that the plastic zone presents itself an infinitely long cylinder ($r_f \leqslant r \leqslant r_c$, $0 \leqslant \theta \leqslant 2\pi$, $-\infty < z < +\infty$; $r_c \leqslant r_m$) and spreads with developing thermal loading, i.e. with decreasing matrix temperature, into the matrix coating. The equation of the current elastic-plastic boundary could be then written in the form $r_c = r_c(T_m)$.

Finally, it will be assumed that the crack length $2l$ is small compared with the radius r_m and that the crack itself is situated relatively far-away from the fibre. This implies the possibility of neglecting the effect of the crack on the stress-strain state within the matrix region just surrounding the fibre where the plastic deformation process actually develops. Then, the latter could be viewed as an axisymmetric one as in the case of an uncracked unit cell.

With this in mind the thermal stress field within the cracked matrix phase could be considered to be a superposition of the following two fields. The first one is the elastic-plastic stress field for the uncracked unit cell while the second one is the field resulting from the presence of a Dugdale type crack.

The plastic deformation process

The model of the plastic deformation process proposed in [16] will be generalized in the present paper with regard to the considered thermal loading problem. The possibility for such a generalization follows from the fact that this model is based in general upon certain effects of the fibre-reinforcement which are common for both the isothermal [16] and the here considered thermal problem. Firstly, to these effects belongs the so-called „shrinkage effect", i.e. the appearance of compressive radial stresses over the fibre-matrix interface. One comes up with this effect provided relation (2) is satisfied which is actually the here considered case. Secondly, in accordance with the elastic solution [18] the fibre acts as a stress concentrator. Because of the local nature of this stress concentration effect one could expect that especially for the considered composites with low fibre volume fractions intensive plastic deformation and even fracture processes may develop within the immediate surrounding of a fibre whereas at a certain distance from the fibre-matrix interface the matrix material may deform still elastically. Thirdly, it is well-known from experimental observations that because of the strengthening effect of the fibre the behaviour of the composite „in the fibre direction" is rather elastic-like than perfectly-plastic. This implies the reasonable assumption that the fibre, consisting itself of linearly elastic material with a high stiffness, contributes due to the assumed perfect fibre-matrix contact to the development of a relatively large elastic part ε_z^e of the total axial strain ε_z within the plastificated region and prevents thus the occurrence of a corresponding large plastic part ε_z^p. In other words in the course of the deformation process one should permanently account for the current elastic part of the axial strain. It is obvious that for the considered regime of thermal loading both the ε_z^e- and ε_z^p-strains should be monotonously increasing in absolute value functions in dependence of the absolute value of the matrix temperature. A reasonable restriction concerning the behaviour of the ε_z^e-strain is associated with the assumption that the matrix material is a perfectly-plastic one and its elastic response is thus limited. One should expect correspondingly that for the considered unit cell and type of loading there exists a certain critical value $\overset{*}{\varepsilon}{}_z^e$ of ε_z^e such that upon reaching this value the current increments of the ε_z^e-strain become negligible with respect to the corresponding increments of ε_z^p. Due to the concentration effect of the fibre this critical value $\overset{*}{\varepsilon}{}_z^e$ should be first achieved over the fibre-matrix interface.

The account for the just introduced limiting characteristic $\overset{*}{\varepsilon}{}_z^e$ implies the following natural description of the plastic deformation process. The plastic deformations appear first over the fibre-matrix interface and the plastic zone $r_f \leqslant r \leqslant r_c$ spreads consequently into the matrix phase. Within this zone both the ε_z^e- and ε_z^p-strains increase simultaneously up to the instant when $\varepsilon_z^e|_{r=r_f} = \overset{*}{\varepsilon}{}_z^e$. At this instant a second plastic zone $r_f \leqslant r \leqslant R_c$ where $R_c < r_c$ appears within which the relation $\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$ holds true while the ε_z^p-strain further increases. The second plastic zone also spreads into the matrix phase having the first one, which occupies now the region $R_c \leqslant r \leqslant r_c$, at its front $r = R_c$.

The model of the plastic deformation process just considered implies a simple possible scheme of an approximate analysis of the elastic-plastic behaviour of the uncracked composite cell.

Analysis of the uncracked unit cell

In accordance with the standard assumption of the plasticity theory the total axial strain at each instant of the plastic deformation process is a sum of an elastic and a plastic part. As usually it will be assumed that the matrix material is plastically incompressible which implies the validity of the following relation within the plastic zone

$$\varepsilon^e_{z_{\text{total}}} = e_z^e + \frac{1}{3}\,\varepsilon^{(\text{str})} + \frac{1}{3}\,\varepsilon^{(\text{temp})}, \tag{3}$$

where e_z^e is the deviatoric axial elastic strain and $\varepsilon^{(\text{str})}$ and $\varepsilon^{(\text{temp})}$ are the relative volume changes associated with the thermal stresses and the thermal expansion respectively, i.e.

$$\varepsilon^{(\text{str})} = \frac{1-2\nu_m}{E_m}\,(\sigma_r+\sigma_\theta+\sigma_z), \tag{4}$$

$$\varepsilon^{(\text{temp})} = 3\alpha_m T_m. \tag{5}$$

In equation (4) as well as in the following analysis σ_i, $i = r, \theta, z$ denote the normal stresses within the plastic zone.

It will be further accepted that the stresses and the elastic strains are as usually related by the Hooke's law so that one has in particular

$$\sigma_z = E_m \varepsilon_z^e + \nu_m(\sigma_r + \sigma_\theta). \tag{6}$$

In equation (6) as well as in the rest of the paper the notation

$$\varepsilon_z^e = e_z^e + \frac{1}{3}\,\varepsilon^{(\mathrm{str})}, \tag{7}$$

is used so that ε_z^e means (cf. equation (3)) the part of the axial elastic strain due to the thermal stresses.

Let the matrix material obey the von Mises' yield condition, i.e. let the stresses σ_i, $i = r, \theta, z$ satisfy the relation

$$(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 = 2\sigma_y^2. \tag{8}$$

Upon substituting for the σ_z-stress from equation (6) into the latter relation one obtains

$$\left(\frac{\sigma_\theta - \sigma_r}{2}\right)^2 + \left(\frac{\sigma_r + \sigma_\theta}{2} - \frac{E_m \varepsilon_z^e}{1 - 2\nu_m}\right)^2 \frac{(1 - 2\nu_m)^2}{3} = \frac{\sigma_y^2}{3}. \tag{9}$$

Now it is a matter of simple computation to show that equation (9) is identically satisfied provided the stresses are presented in the form

$$\left.\begin{matrix}\sigma_r\\ \sigma_\theta\end{matrix}\right\} = \frac{E_m \varepsilon_z^e}{1 - 2\nu_m} + \frac{\sigma_y}{\sqrt{3}\sin\phi}\cos(\omega \pm \phi), \tag{10}$$

where the notations are used

$$\sin\omega = \frac{\sigma_\theta - \sigma_r}{2} \Big/ \frac{\sigma_y}{\sqrt{3}}, \tag{11}$$

$$\operatorname{cotan}\phi = \frac{\sqrt{3}}{1 - 2\nu_m}. \tag{12}$$

Equations (10) reflect the implicit assumption that $\sigma_\theta \geqslant \sigma_r$ which implies in accordance with equation (11) that $0 \leqslant \omega \leqslant \pi$.

Substituting now for the stresses σ_r and σ_θ from equations (10) into the equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{13}$$

one obtains the equation

$$\frac{E_m}{1 - 2\nu_m}\frac{d\varepsilon_z^e}{dr} - \frac{\sigma_y}{\sqrt{3}\sin\phi}\sin(\omega + \phi)\frac{d\omega}{dr} - \frac{2\sigma_y}{\sqrt{3}}\frac{\sin\omega}{r} = 0, \tag{14}$$

where ε_z^e is an unknown function of the radial coordinate r and therefore the integration of equation (14) cannot be performed. But an approximate solution of equation (14) can be obtained which is valid at least within the immediate surrounding of the fibre.

This solution is based upon the assumption that within this region the elastic parts of the ε_r- and ε_θ-strain components due to the thermal stresses are negligible with respect to the corresponding plastic parts ε_r^p and ε_θ^p. This implies the relation

$$\varepsilon_z^e = \varepsilon^{str}. \tag{15}$$

Equation (15) together with equations (4), (6) and (10) gives now

$$\frac{\sigma_y(1+\nu_m)}{E_m}\cos\omega = -\varepsilon_z^e \tag{16}$$

The latter relation applies as assumed in a thin layer surrounding the fibre and over the fibre-matrix interface $r = r_f$ in particular where the condition $\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$ is first achieved. The corresponding value $\omega^* \equiv \omega(\overset{*}{\varepsilon}{}_z^e)$ of the angle ω follows from equation (16) to be

$$\omega^* = \arccos\left[-\frac{E_m \overset{*}{\varepsilon}{}_z^e}{\sigma_y(1+\nu_m)}\right] \tag{17}$$

According to the model of the plastic deformation process proposed in section 3 above a further increase in thermal loading which corresponds to a further decrease in the matrix temperature T_m results in the appearance of a second plastic zone $r_f \leqslant r \leqslant R_c$ over the outer boundary of which equation (17) is valid. Finally, assuming that for the considered unit cell and the given scheme of loading the quantity $\overset{*}{\varepsilon}{}_z^e$, respectively ω^* (cf. equation (17)) is approximately constant and introducing the angle ω_{R_c} as

$$\omega_{R_c} = \omega(R_c), \tag{18}$$

one obtains

$$\omega_{R_c} = \omega^* \tag{19}$$

where ω^* is a constant now.

Now the latter assumption makes equation (14) integrable within the whole second plastic zone $r_f \leqslant r \leqslant R_c$. The result of this integration with the boundary condition $\omega|_{r=R_c} = \omega_{R_c}$ reads

$$\frac{R_c^2}{r^2} = \frac{\sin\omega}{\sin\omega_{R_c}} \exp\left[\frac{\sqrt{3}}{1-2\nu_m}(\omega-\omega_{R_c})\right]. \tag{20}$$

Moreover, it could be easily verified that the set of equations (6), (10), (17), (19) and (20) defines the stress state entirely within the second plastic zone $r_f \leqslant r \leqslant R_c$, where R_c has still to be determined.

The stress state defined above allows certain important conclusions concerning the fracture behaviour of the considered unit cell. To this end we consider the shrinkage effect again. It is clear from most general positions that this effect is due to the difference in the lateral contraction of the fibre and matrix materials. Because of the plastic incompressibility of the matrix material this difference should be expected to increase in the course of the deformation process. In other words developing plastic deformations should further contribute to the shrinkage effect or, equivalently, the radial stress $\sigma_r|_{r=r_f}$ acting over the fibre-matrix interface should decrease with increasing loading, i.e. with decreasing temperature of the matrix phase. The latter means in accordance with equations (10) that the angle ω_{r_f} should increase in the course of the deformation process remaining obviously larger than the angle ω^*. Moreover, equations (10) show that there exists a natural limitation of the shrinkage effect in the sense that this effect achieves its maximum at a value of $\omega_{r_f} = \pi - \phi$.

The value R_c^* of the radius R_c at this instant, that is

$$R_c^* = R_c|_{\omega_{r_f}} = \pi - \phi, \tag{21}$$

follows from equation (20) to be

$$R_c^{*2} = r_f^2 \frac{\sin\phi}{\sin\omega_{R_c}} \exp\left[\frac{\sqrt{3}}{1-2\nu_m}(\pi - \phi - \omega_{R_c})\right]. \tag{22}$$

The model of the process applied here leads thus to the conclusion that further decrease in $\sigma_r|_{r=r_f}$ as well as increase in R_c is impossible. Further, it would be of interest to examine the velocity field corresponding to this limiting state of the plastic deformation process within a unit cell.

To this regard the known concept of the associated flow rule will be applied with the yield function (9) serving as a plastic potential. Simple computations show that in accordance with this concept and the plastic incompressibility condition the plastic strain rates ξ_i, $i = r, \theta, z$ satisfy the relation

$$\xi_z = -\xi_\theta \frac{\Sigma_\theta + \Sigma_r}{\Sigma_\theta}, \tag{23}$$

within the second plastic zone where

$$\left.\begin{matrix}\Sigma_r \\ \Sigma_\theta\end{matrix}\right\} = \frac{\sigma_y}{\sqrt{3}} \sin(\phi \mp \omega). \tag{24}$$

It is easily observed from the latter equations that $\xi_r|_{r=r_f} \to +\infty$ when $\omega_{r_f} \to \pi - \phi$. This result means physically that at this state free plastic flow tends to take place within a thin layer immediately surrounding the fibre. The behaviour of the composite at this state will obviously depend upon the interaction between this tendency and the strengthening effect of the fibre which tends itself to prevent the occurrence of such a singular velocity field. The very nature of these two competing effects implies the reasonable assumption that their interaction results in the occurrence of shearing stresses over the fibre-matrix interface. Moreover, these shearing stresses should be equal for obvious reasons to the shear yield stress $\tau_y = \sigma_y/\sqrt{3}$ of the matrix material.

Let τ_s be the shear strength of the fibre-matrix interface. If $\tau_s \leqslant \tau_y$ then the very reaching of the considered critical state will obviously result in the immediate failure of the fibre-matrix interface by the so-called debonding effect. If, on the contrary, $\tau_s > \tau_y$, then the known mechanism of fibres pull-out (see, for example [9]) will develop, most probably together with a process of fibre breaking.

Plastic zone size and associated problems

In order to close the solution of the problem for the uncracked unit cell one should complete the results of the previous section with the temperature dependence of the radius of the plastic zone. Moreover, when dealing with a given composite material one should specify the actual value of $\overset{*}{\varepsilon}{}_z^e$ which should be used in the computations.

The model of the plastic deformation process proposed above implies a simple approach to the latter problem. Starting point for this approach is the additional assumption that the first plastic zone presents itself a thin layer and thus one may consider the relation $R_c = r_c$ to hold approximately true. This assumption appears as acceptable one for the following reasons. Firstly, because of the local nature of the fibre concentration effect and since a low fibre volume fraction composite is considered. Therefore, both the R_c- and r_c-radii should be small compared with the value of r_m. Secondly, because of the low resistance of the matrix material with respect to the occurrence of intense plastic deformation such as the deformations within the second plastic zone are. Thus, one may expect that the transition zone between the elastically deformed matrix region and the second plastic zone is really a thin one. If so, then the first plastic zone could be simply considered to play the role of an elastic-plastic boundary, the latter having the form of a thin layer. Further, because of the thin layer shape of the elastic-plastic boundary a softened version of fulfillment of the standard elastic-plastic transition conditions of continuity of stresses and displacements could be applied, namely the following.

Firstly, because of the layer thinness one should not expect a substantial change of the radial stress within the layer itself which implies the relation

$$\sigma_r|_{r=R_c} = \sigma_r^e|_{r=R_c}, \tag{25}$$

where σ_i^e, $i = r, \theta, z$ are the stresses acting within the elastic region $R_c \leqslant r \leqslant r_m$ of the matrix phase. Secondly, these stresses should satisfy the yield condition, equation (8), over the elastic-plastic boundary, that is

$$[(\sigma_r^e-\sigma_\theta^e)^2+(\sigma_\theta^e-\sigma_z^e)^2+(\sigma_z^e-\sigma_r^e)^2]|_{r=R_c} = 2\sigma_y^2. \tag{26}$$

In accordance with the general form of the elastic solution of the problem [11, 18] and the results of the previous section one may present the latter equations in the form

$$\frac{E_m}{1+\nu_m} C\left(\frac{1}{r_m^2} - \frac{1}{R_c^2}\right) = \frac{E_m \overset{*}{\varepsilon}{}_z^e}{1-2\nu_m} + \frac{\sigma_y}{\sqrt{3}\sin\phi}\cos(\omega_{R_c}+\phi), \tag{27}$$

$$3\frac{C^2}{R_c^4} = \frac{\sigma_y^2(1+\nu_m)^2}{E_m^2} - \left[\frac{C(1-2\nu_m)}{r_m^2} - (1+\nu_m)(\varepsilon_z - \alpha_m T_m)\right]^2, \tag{28}$$

where the constant C has to be determined actually.

The remaining elastic-plastic transition conditions could be now viewed as satisfied as well in this way that the corresponding stresses and displacements change continuously within the layer between their values on its „elastic" and „plastic" surface. Thus, the equations (27) and (28), respectively, present the just mentioned softened version of the elastic-plastic transition conditions.

If solved for the unknowns R_c and C the set of equations (27) and (28) implies as a matter of fact the temperature dependence of both R_c and C, respectively, in the form

$$R_c = R_c(T_m, \varepsilon_z; \overset{*}{\varepsilon}{}_z^e, E_m, \nu_m, \alpha_m, r_m, \sigma_y), \tag{29}$$

$$C = C(T_m, \varepsilon_z; \overset{*}{\varepsilon}{}_z^e, E_m, \nu_m, \alpha_m, r_m, \sigma_y), \tag{30}$$

where ε_z itself is a still unknown function of the matrix temperature T_m.

It is important to mention at this point that with R_c and C once determined from the set of equations (27) and (28) one may consider the axial stresses σ_z and σ_z^e acting within the plastic and elastic regions of the matrix phase, respectively, to be known functions of the same parameters as those in the presentations (29) and (30). The corresponding expressions for these stresses can be easily given by

$$\sigma_z = \frac{1}{1-2\nu_m}(\mathrm{E}_m \overset{*}{\varepsilon}{}_z^e + 2\sigma_y \cos\omega), \quad r_f \leqslant r \leqslant R_c, \tag{31}$$

$$\sigma_z^e = \mathrm{E}_m \varepsilon_z + \frac{2\nu_m \mathrm{E}_m C}{(1+\nu_m) r_m^2}, \quad R_c \leqslant r \leqslant r_m, \tag{32}$$

where both R_c and C can be considered now as known functions of the form given by the equations (29) and (30), respectively.

Further, it is a matter of a simple verification that upon satisfying the continuity condition for the radial stresses over the fibre-matrix interface, i.e. the equation

$$\sigma_r^m|_{r=r_f} = \sigma_r^f|_{r=r_f}, \tag{33}$$

one may construct the expressions for the stresses σ_i^f, $i = r, \theta, z$ acting within the fibre. Thereby the expression for the axial stress σ_z^f reads

$$\sigma_z^f = \mathrm{E}_f \varepsilon_z + 2\nu_f \left[\frac{\mathrm{E}_m \overset{*}{\varepsilon}{}_z^e}{1-2\nu_m} + \frac{\sigma_y}{\sqrt{3}\sin\phi}\cos(\omega_{r_f}+\phi)\right], \quad 0 \leqslant r \leqslant r_f, \tag{34}$$

where the value of ω_{r_f} follows from equation (20) with $r = r_f$ and with the quantity R_c given in the form of equation (29).

It is easily observed that the axial stresses as presented by the equations (31), (32) and (34) can be considered now as known functions of T_m and ε_z and the remaining parameters of the problem, i.e. $\overset{*}{\varepsilon}{}_z^e$, σ_y and $E_i, \nu_i, \alpha_i, r_i$ where $i = f, m$. These stresses have to satisfy the equilibrium condition for the forces, acting in the axial direction, which in our self-stress problem is given by the following condition of self—equilibrium

$$\sigma_z^f \cdot r_f^2 = 2\int_{r_f}^{R_c} \sigma_z r dr + (r_m^2 - R_c^2)\sigma_z^e. \tag{35}$$

By substituting for R_c from equation (29) into equation (35) leads to a relation of the form

$$\varepsilon_z = \varepsilon_z(T_m; \overset{*}{\varepsilon}{}_z^e, \mathrm{E}_i, \nu_i, \alpha_i, r_i, \sigma_y), \tag{36}$$

where $i = f, m$. Equation (36) represents in fact the equation of the theoretical ε_z versus T_m curve in the framework of the proposed model for the considered two-phase material.

Upon substituting for ε_z from equation (36) into equation (29) one obtains the desired dependence of the plastic zone radius R_c on the matrix temperature T_m. This dependence is obviously of the form

$$R_c = R_c(T_m; \overset{*}{\varepsilon}{}_z^e, \mathrm{E}_i, \nu_i, \alpha_i, r_i, \sigma_y), \tag{37}$$

where again $i = f, m$.

Note that by applying equation (37) to the critical state of the unit cell, i.e. if $R_c = R_c^*$ (cf. equation (22)), one obtains the critical temperature T_m^* at which one of the failure modes of the fibre-matrix interface described in section 4 occurs. This critical temperature appears to be of the form

$$T_m^* = T_m^*(R_c^*, \overset{*}{\varepsilon}{}_z^e, E_i, \nu_i, \alpha_i, r_i, \sigma_y); \quad i = f, m. \tag{38}$$

Equation (38) implies itself a simple criterion of failure of the fibre-matrix interface of the form

$$T_m = T_m^* \tag{39}$$

Now it is easily observed that the actual value of $\overset{*}{\varepsilon}{}_z^e$ can be determined by means of a comparison of the theoretically predicted $\varepsilon_z(T_m)$-curve, equation (36), with a corresponding curve obtained experimentally for the considered composite material.

A simpler approach to the problem consists in the determination of $\overset{*}{\varepsilon}{}_z^e$ from equation (38) provided the T_m^*- and R_c^*-values in this equation are the values which have been observed experimentally.

The scheme described above does not imply a closed form solution for the quantities R_c and $\overset{*}{\varepsilon}{}_z^e$ but the associated numerical treatment of the problem is not very complicated. With this in mind one may consider that the whole problem concerning the elastic-plastic behaviour of the uncracked unit cell has been solved completely.

The cracked unit cell

It should be remembered in the following that as accepted in section 2 the thermal stress field within the cracked matrix phase can be considered as a superposition of an elastic-plastic stress field for the uncracked unit cell and a corrective stress field caused by the presence of a crack situated along the segment $R_l \leqslant r \leqslant R_r$, $\theta = 0$ of the symmetry line of the cross section of the unit cell. Now the first stress field is known from the preceding analysis, sections 4 and 5. The second stress field will be examined as already mentioned in the framework of the Dugdale crack model [17].

The thermal stress field in the uncracked elastically deformed matrix region is given by the following expressions

$$\left.\begin{matrix}\sigma_r^e \\ \sigma_\theta^e\end{matrix}\right\} = \frac{E_m}{1+\nu_m} C\left(\frac{1}{r_m^2} \mp \frac{1}{r^2}\right), \qquad R_c \leqslant r \leqslant r_m, \tag{39a}$$

$$\sigma_z^e = E_m(\varepsilon_z - \alpha_m T_m) + \frac{2\nu_m E_m}{1+\nu_m} C \frac{1}{r_m^2},$$

where $C = C(T_m, \ldots)$ and $R_c = R_c(T_m, \ldots)$ can be considered as known functions of the temperature T_m in accordance with the results of the previous section.

The corrective stress firld σ_{ij}^c, $i, j = r, \theta, z$ can be obtained from the solution of the following mixed boundary-value problem for a Dugdale type crack with an actual length $2l = r_r - r_l$ and a fictitious length $2L = R_r - R_l$ (cf. Fig. 1 for notation)

$$\sigma_{\theta\theta}^c(r, 0) = \begin{cases} -\sigma_\theta^e, & r_l \leqslant r \leqslant r_r, \\ -\sigma_\theta^e + \sigma_y, & R_l \leqslant r \leqslant r_l \quad \text{and } r_r \leqslant r \leqslant R_r, \end{cases} \tag{40}$$

$$\sigma^c_{r\theta}(r, 0) = 0, \quad \theta = 0, \forall r, \tag{41}$$

$$u_\theta(r, 0) = 0, \quad \theta = 0, \quad R_r \leqslant r \leqslant R_l. \tag{42}$$

The boundary-value problem (40) - (42) has been analyzed in detail by HERRMANN [11]. Applying the results obtained in [11] together with those presented above one immediately obtains the desired plastic zone lengths s_l and s_r (cf. Fig. 1) for our case from the solutions of the equation

$$\frac{E_m C}{2(1+\nu_m) r_m^2}\left\{1 + \frac{r_m^2[r_0 \pm (l+s)]}{[r_0^2-(l+s)^2]^{3/2}}\right\} - \frac{\sigma_y}{\pi}\arccos\frac{l}{l+s} = 0, \tag{43}$$

where s is the unknown and $r_0 = (r_r + r_l)/2$. The quantities s_l and s_r correspond to the upper and lower signs, respectively, in the brackets in equation (43).

Upon solving equation (43) one obtains the plastic zone lengths as functions of the temperature T_m and the actual crack length $2l$, i.e. the relations

$$s_i = s_i(T_m, l), \quad i = r, l. \tag{44}$$

It is well understood that each of the quantities s_i, $i = r, l$ depends in addition on the remaining parameters of the problem as well so that the representations (44) are actually schematic ones. They imply in an obvious way the relations

$$R_i = R_i(T_m, l), \quad i = r, l. \tag{45}$$

The equations (45) present the dependence of the positions of the left and the right tip of the imaginary crack on the current matrix temperature. That dependence could be now considered as known from the solution of equation (43). The latter solution could be itself obtained by means of a numerical treatment [11].

Crack and cell behaviour

As already accepted (cf. equation (38)) let T_m^* be the value of the matrix temperature at which one of the two modes of failure of the fibre-matrix interface (cf. section 4) occurs and let R_l^* be the value of R_l corresponding to this value of the temperature (cf. equation (45)). Then R_l^* defines in fact the position of the left plastic zone tip at the instant of failure of the fibre-matrix interface. Further, let $\bar{R}_l$ be the value of R_l at which the crack begins to grow in accordance with a certain crack growth criterion and let $\bar{T}_m$ be the corresponding matrix temperature, i.e.

$$\bar{R}_l = R_l(\bar{T}_m, l). \tag{46}$$

Then equation (46) corresponds to the equation (45) but now applied to the instant of crack growth initiation. It is assumed implicitly that due to the fibre concentration effect the crack will start to propagate from its left tip toward the fibre.

It follows from the whole scheme of analysis that the values of both quantities R_l^* and $\bar{R}_l$, respectively, appear as specific ones for each given cracked unit cell or, equivalently, for a given composite material. Thereby both values could be defined from the present analysis provided the corresponding crack growth criterion is given. The latter concerns obviously the determination of $\bar{R}_l$. Once evaluated for a certain unit cell with a given crack configuration these R_l^*- and $\bar{R}_l$-values imply immediately the following evident conclusions concerning the behaviour of the cracked unit cell. If $R_l^* > \bar{R}_l$ then the fibre-matrix interface fails while the crack is still in rest. If on the contrary $R_l^* < \bar{R}_l$ then the crack growth initiation precedes the failure of the fibre-matrix interface. In this case the whole scheme of analysis remains further valid (up to the possible failure of the interface) provided the crack propagates quasi-statically and $2l$ is its current length upon which both R_l^* and $\bar{R}_l$ depend. Finally, if $R_l^* = \bar{R}_l$ then the fibre-matrix interface fails simultaneously with the initiation of the crack propagation. The further behaviour of the crack and the unit cell in this case as well as in the first one where $R_l^* > \bar{R}_l$ needs a new approach since the present considerations are based upon the assumption of a perfect fibre-matrix contact.

It should be mentioned that these simple conclusions are valid under the assumptions made earlier that both the crack length $2l$ and the plastic zone radius R_c are small compared with the geometrical quantities r_m and r_0, respectively. That means the crack should not influence the stress state within the plastic zone $r_f \leqslant r \leqslant R_c$ where the solution of section 4 is thus expected to apply. Moreover, it has been shown in [19] that the approximate anaytical elastic solution of the considered thermal crack problem in a composite unit cell lobtained in [11] remains still valid even if the restrictions concerning the quantities $2l$, r_0, r_m and r_f (the latter quantity plays the role of R_c in the elastic case) are somehow violated. One may consequently expect that the results of the present elastic-plastic analysis will also remain valid when the restrictions mentioned above are somehow softened since the values of R_i, $i = r, l$ used in our considerations are actually obtained from the same elastic solution [11]. If so, then one comes easily up with a couple of further implications of the analysis concerning the crack and cell behaviour.

Let $\tilde{T}_m$ be the temperature at which both plastic zones, i.e. the annulus $r_f \leqslant r \leqslant R_c$ and the segment $R_l \leqslant r \leqslant r_l$ join each other, and let $\tilde{R}_c = R_c(\tilde{T}_m)$ and $\tilde{R}_l = R_l(\tilde{T}_m, l)$ be the corresponding values of the quantities R_c and R_l, respectively. Further, let us assume the validity of the relations $\tilde{T}_m > T_m^*$ and $\tilde{T}_m > \bar{T}_m$ (note that the temperatures T_m^*, $\tilde{T}_m$ and $\bar{T}_m$ are negative) for the considered unit cell. That means that the two plastic zones meet each other before the conditions for the failure of the fibre-matrix interface and for crack growth initiation are fulfilled. Upon reaching this instant, that is with the two plastic zones adjoined, the behaviour of the cracked unit cell will depend essentially on the interaction between the plastic mechanism of failure of the entirely plastificated segment $r_f \leqslant r \leqslant r_l$, $\theta = 0$ and the brittle mechanism of crack growth at the right crack tip $r = r_r$. A possible approach to this problem could be based upon the application of the rigid-plastic body model and the limit load concept associated with this model (see, for example [20]). When applying the latter concept to the plastificated segment $r_f \leqslant r \leqslant r_l$, $\theta = 0$ one may expect that further thermal loading, i.e. further decrease of T_m, will result in the activation of the crack propagation mechanism at the right crack tip.

Another case of interest is the one for which $\tilde{T}_m \leqslant \bar{T}_m$ and $\tilde{T}_m > T_m^*$. In this case both plastic zones join each other again but the segment $R_l \leqslant r \leqslant r_l, \theta = 0$ presents now the plastic zone at the left tip of the running crack. Depending upon the plastic zone thickness $d = R_c - r_f$ and the crack velocity the crack may stop before reaching the elastic-plastic boundary or at the boundary or may traverse partially or entirely the plastificated annulus around the fibre. Depending in addition on the fibre-matrix contact the crack may stop at the fibre-matrix interface in order to traverse it or to create an interface crack. The investigation of all these possibilities, being a problem of definite interest, is associated with considerable difficulties arising from the necessity of solving boundary-value problems for cracks partially situated within plastificated regions as well as of applying reasonable criteria of crack propagation and arrest.

Concluding remarks

The analysis presented above implies certain definite conclusions concerning the behaviour of a cracked unit cell of a fibre-reinforced composite material under the conditions of thermal loading. The analysis could be easily transformed to the more general case $T_f \neq 0$ if the temperature difference $\hat{T} = T_m - T_f$ will actually play the role of the quantity T_m used in the preceding calculations. In that case the linear coefficient of thermal expansion α_f will influence the processes of plastification and fracture as well.

The model of the plastic deformation process proposed leads to both closed form results and to a relatively simple procedure concerning the numerical treatment of the problem on the whole. The analysis shows that the entire solution of the considered problem is associated with the necessity of directed experimental investigations concerning the determination of the specific measure of elastic response $\overset{*}{\varepsilon}{}_z^e$ for the fibrous composite materials.

Acknowledgement

The support of the Alexander von Humboldt Foundations for one of the authors (I.M.M.) is gratefully acknowledged.

References

1. C. W. SMITH, *Limitations of Fracture Mechanics as Applied to Composites*, In C. T. Herakovich (Ed.). Inelastic Behavior of Composite Materials, AMD-vol. 13, ASME, New York, pp. 157 - 175, 1975.
2. R. HILL, *Theory of Mechanical Properties of Fibre-Strengthened Materials*, II. Inelastic Behaviour, J. Mech. Phys. Solids, vol. 12, pp. 213 - 218, 1964.
3. A. J. M. SPENCER, *A Theory of The Failure of Ductile Materials Reinforced by Elastic Fibres*, Int. J. Mech. Sciences, vol. 7, pp. 197 - 209, 1965.
4. J. F. MULHERN, T. G. ROGERS and A. J. M. SPENCER, *A Continuum Model for Fibre-Reinforced Plastic Materials*, Proc. Royal Society, A 301, pp. 473 - 492, 1967.
5. G. A. COOPER and M. R. PIGGOTT, *Cracking and Fracture of Composites*, In D. M. R. Taplin (Ed.), Fracture 1977, University of Waterloo Press, vol. 1, pp. 557 - 605, 1977.
6. C. T. HERAKOVICH (Ed.), *Inelastic Behavior of Composite Materials*, AMD-vol. 13, ASME, New York, pp. 157 - 175, 1975.
7. A. J. M. SPENCER, *Deformations of Fibre-Reinforced Materials*, Clarendon Press, Oxford, 1972.
8. I. M. KOPIOV and A. S. OVCINSKIJ, *Fracture of Fibre-Reinforced Metals*, Nauka, Moscow, 1977 (in Russian).

9. M. R. PIGGOTT, *Load Bearing Fibre Composites*, Pergamon Press, Oxford, 1980.
10. K. HERRMANN, *Self-Stress Fracture in a Thermoelastic Two-Phase Medium*, Mech. Research Communications, vol. 2, pp. 85 - 90, 1975.
11. K. HERRMANN, *Interaction of Cracks and Self-Stresses in a Composite Structure*, In J. W. Provan (Ed.), SM Study No. 12 „Continuum Models of Discrete Systems", University of Waterloo Press, pp. 313 - 338, 1978.
12. K. HERRMANN, *Quasistatic Thermal Crack Growth in the Viscoelastic Matrix Material of a Brittle Fiber Reinforced Unit Cell*, Mech. Research Communications, vol. 8, pp. 97 - 104, 1981.
13. K. HERRMANN and A. FLECK, *Thermal Fracture in Compound Materials*, In D. M. R. Taplin (Ed.), Fracture 1977, University of Waterloo Press, vol. 3, pp. 1047 - 1054, 1977.
14. K. HERRMANN, H. BRAUN and P. KEMENY, *Comparison of Experimental and Numerical Investigations Concerning Thermal Cracking of Dissimilar Materials*, Intern. J. Fracture, vol. 15, R 187 - 190, 1979.
15. H. BRAUN and K. HERRMANN, *Analysis of Thermal Cracking of Unidirectionally Reinforced Composite Structures in the Micromechanical Range*, In D. Francois (Ed.), Advances in Fracture Research, Pergamon Press, vol. 1, pp. 485 - 493, 1981.
16. K. HERRMANN and I. M. MIHOVSKY, *Plastic Behaviour of Fibre-Reinforced Composites and Fracture Effects*, Proccedings of the Fourth National Congress of Theoretical and Applied Mechanics, Varna/Bulgaria, vol. 1 pp. 431 - 436, 1981.
17. D. S. DUGDALE, *Yielding of Steel Sheets Containing Slits*, J. Mech. Phys. Solids, vol. 8, pp. 100 - 104, 1960.
18. K. HERRMANN, *Über Eigenspannungen im diskontinuierlich inhomogenen Festkörper*, in K. Schröder (Ed.), Beiträge zur Spannungs- und Dehnungsanalyse, Akademie Verlag, Berlin, vol. 6, pp. 21 - 52, 1970.
19. H. BRAUN, A. FLECK and K. HERRMANN, *Finite Element Analysis of a Quasistatic Crack Extension in a Unit Cell of a Fiber-Reinforced Material*, Intern. J. Fracture, vol. 14, R 3 - 6, 1978.
20. L. M. KACHANOV, *Foundations of the Theory of Plasticity*, North Holland, Amsterdam, 1971.

Резюме

РАЗМЕР ПЛАСТИЧЕСКОЙ ЗОНЫ ТРЕЩИНЫ, ТИПА ДАГДЕЙЛЯ, В ПРЕДВАРИТЕЛЬНО НАПРЯЖЕННОЙ ДВУХФАЗНОЙ СРЕДЕ, С ЧАСТИЧНО ПЛАСТИЧЕСКОМ МАТЕРИАЛОМ МАТРИЦЫ

Предлагается модель пластической деформации касающийся одной ячейки композита усиленного волокнами. Мы исследовали процесс деформации ячейки при учете возможного разрушения в месте соединения матрицы и волокна. При учете модели Дагдейля мы сделали выводы касающиеся термического возраста радиальных трещин в матрице.

Streszczenie

ZASIĘG STREFY UPLASTYCZNIENIA SZCZELIN TYPU DUGDALE'A WE WSTĘPNIE NAPRĘŻONYM OŚRODKU DWUFAZOWYM Z CZĘŚCIOWO UPLASTYCZNIONYM MATERIAŁEM MATRYCY

Proponujemy model odkształcenia plastycznego dotyczący pojedynczej komórki kompozytu wzmocnionego włóknami. Zbadany został proces odkształcenia komórki przy uwzględnieniu możliwego mechanizmu pękania w miejscu połączenia matrycy i włókna.

Przy uwzględnieniu modelu Dugdale'a wyciągnięto wnioski dotyczące termiczengo wzrostu szczelin promieniowych w matrycy.

Praca została złożona w Redakcji dnia 8 czerwca 1983 roku