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IMPLICIT SCHEME OF THE FINITE DIFFERENCE METHOD FOR THE SECOND-ORDER DUAL PHASE LAG EQUATION

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The second-order dual phase lag equation (DPLE) as a mathematical model of the microscale heat transfer is considered. It is known that the starting point determining the final form of this equation is the generalized Fourier law in which two positive constants (the relaxation and thermalization times) appear. Depending on the order of the generalized Fourier law expansion into the Taylor series, different forms of the DPLE can be obtained. As an example of the problem described by the second-order DPLE equation, thermal processes proceeding in the domain of a thin metal film subjected to a laser pulse are considered. The numerical algorithm is based on an implicit scheme of the finite difference method. At the stage of numerical modeling, the first, second and mixed order of the dual phase lag equation are considered. In the final part of the paper, examples of different solutions are presented and conclusions are formulated.

Keywords: microscale heat transfer, dual phase lag model, implicit scheme of finite difference method

1. Introduction

The Fourier heat conduction model is based on the assumption of instantaneous propagation of the thermal wave in the domain considered. Intuitively, this approach seems to be incorrect, but it has worked for solving a number of macroscopic heat conduction problems. However, it turned out that for certain non-typical materials with a complex internal structure, the Fourier model is insufficient (Roetzel *et al.*, 2003). Even more, deviations from the real course of the process can be seen in the case of microscale heat transfer.

It is obvious that accumulating enough energy to transfer to the nearest neighborhood would take time in the process of heat transfer (Zhang, 2007). So, the lag time of the heat flux in relation to the temperature gradient referred to as "a relaxation time" was introduced by Cattaneo (1948) and Vernotte (1958), and the appropriate energy equation (a hyperbolic PDE) became known as the Cattaneo-Vernotte equation. In the recent years, the heat conduction model in which two delay times appear has become more and more popular. This model is called the dual-phase lag one (Zhang, 2007; Tzou, 2015). The starting point for considerations is the generalized form of the Fourier law, e.g. (Faghri *et al.*, 2010; Smith and Norris, 2003). Depending on the number of terms in the Taylor series expansion of this law, different forms of the dual phase lag equation (DPLE) can be obtained (see Section 2). The lag times appearing in DPLE are called the relaxation time and the thermalization time. Some simple tasks described by this equation (supplemented by appropriate boundary and initial conditions) can be solved analytically, e.g. (Ciesielski, 2017a; Tang and Araki, 1999; Askarizadeh *et al.*, 2017; Mohammadi-

-Fakhar and Momeni-Masuleh, 2010). However, most of the practical problems have been solved using numerical methods. Examples of such solutions in the field of the microscale heat transfer may be the papers (Majchrzak and Mochnacki, 2014; Ciesielski, 2017b; Dai and Nassar, 2000; Mochnacki and Paruch, 2013; Chen and Beraun, 2001) concerning the first-order DPLE.

The similar problems have been considered for non-homogeneous (multilayered) domains. In this place, the papers (Majchrzak *et al.*, 2009; Qiu *et al.*, 1994; Al-Nimr *et al.*, 2004; Wang *et al.*, 2006, 2008) can be (as the examples) mentioned. The correct form of the boundary conditions between subdomains (here, the macroscopic boundary conditions are often used, which is a significant simplification) can be found in (Ho *et al.*, 2003) while the detailed mathematical considerations were shown in (Majchrzak and Kałuża, 2017). In turn, in the paper (Majchrzak and Mochnacki, 2016), the problem of stability condition (explicit scheme of the FDM) was analyzed.

The numerical solutions concerning the second-order DPLE (based on the finite difference method) are the subject of works prepared by Castro *et al.* (2016) and Deng *et al.* (2017). The similar problems are discussed in the paper presented, but the wider class of equations and the other numerical algorithm are taken into account.

The applications of DPLE in the scope of bioheat transfer will not be discussed here.

2. Dual-phase lag model

The following well known thermal diffusion equation is considered

$$c\frac{\partial T(X,t)}{\partial t} = -\nabla \cdot \mathbf{q}(X,t) + Q(X,t) \tag{2.1}$$

where c is a volumetric specific heat, $\mathbf{q}$ is a heat flux vector, Q is a capacity of the internal volumetric heat source, X, t denote the geometrical co-ordinates and time.

The relationship between the heat flux $\mathbf{q}$ and the temperature gradient ∇T is given in the form of the generalized Fourier law (Zhang, 2007; Smith and Norris, 2003), namely

$$\mathbf{q}(X, t+\tau_q) = -\lambda \nabla T(X, t+\tau_T) \tag{2.2}$$

where λ is thermal conductivity, τ_q and τ_T are the relaxation time and thermalization time, respectively. The relaxation time τ_q is the mean time for electrons to change their energy states, while the thermalization time τ_T is the mean time required for electrons and lattice to reach equilibrium.

Using the Taylor series expansions, the following second-order approximation of formula (2.2) can be taken into account

$$\mathbf{q}(X,t)+\tau_q\frac{\partial \mathbf{q}(X,t)}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2 \mathbf{q}(X,t)}{\partial t^2} = -\lambda\Big[\nabla T(X,t)+\tau_T\frac{\partial \nabla T(X,t)}{\partial t}+\frac{\tau_T^2}{2}\frac{\partial^2 \nabla T(X,t)}{\partial t^2}\Big] \tag{2.3}$$

which means

$$-\mathbf{q}(X,t) = \tau_q\frac{\partial \mathbf{q}(X,t)}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2 \mathbf{q}(X,t)}{\partial t^2}+\lambda\nabla T(X,t)+\lambda\tau_T\frac{\partial \nabla T(X,t)}{\partial t}+\lambda\frac{\tau_T^2}{2}\frac{\partial^2 \nabla T(X,t)}{\partial t^2} \tag{2.4}$$

From equation (2.4) it results that

$$\begin{aligned} -\nabla \cdot \mathbf{q}(X,t) &= \tau_q\frac{\partial[\nabla \cdot \mathbf{q}(X,t)]}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2[\nabla \cdot \mathbf{q}(X,t)]}{\partial t^2}+\nabla[\lambda\nabla T(X,t)] \\ &+\tau_T\frac{\partial\{\nabla[\lambda\nabla T(X,t)]\}}{\partial t}+\frac{\tau_T^2}{2}\frac{\partial^2\{\nabla[\lambda\nabla T(X,t)]\}}{\partial t^2} \end{aligned} \tag{2.5}$$

The last dependence is introduced in to equation (2.1), and then

$$
\begin{aligned}
c\frac{\partial T(X,t)}{\partial t} &= \tau_q \frac{\partial[\nabla\cdot\mathbf{q}(X,t)]}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2[\nabla\cdot\mathbf{q}(X,t)]}{\partial t^2} + \nabla[\lambda\nabla T(X,t)] \\
&+ \tau_T \frac{\partial\{\nabla[\lambda\nabla T(X,t)]\}}{\partial t} + \frac{\tau_T^2}{2}\frac{\partial^2\{\nabla[\lambda\nabla T(X,t)]\}}{\partial t^2} + Q(X,t)
\end{aligned}
\tag{2.6}
$$

Equation (2.1) can also be written as

$$
\nabla\cdot\mathbf{q}(X,t) = -c\frac{\partial T(X,t)}{\partial t} + Q(X,t)
\tag{2.7}
$$

Putting equation (2.7) into (2.6), one obtains

$$
\begin{aligned}
c\frac{\partial T(X,t)}{\partial t} &= \tau_q \frac{\partial}{\partial t}\Big[-c\frac{\partial T(X,t)}{\partial t} + Q(X,t)\Big] + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\Big[-c\frac{\partial T(X,t)}{\partial t} + Q(X,t)\Big] \\
&+ \nabla[\lambda\nabla T(X,t)] + \tau_T \frac{\partial\{\nabla[\lambda\nabla T(X,t)]\}}{\partial t} + \frac{\tau_T^2}{2}\frac{\partial^2\{\nabla[\lambda\nabla T(X,t)]\}}{\partial t^2} + Q(X,t)
\end{aligned}
\tag{2.8}
$$

Assuming the constant value of the volumetric specific heat c, one has

$$
\begin{aligned}
c\Big[\frac{\partial T(X,t)}{\partial t} + \tau_q\frac{\partial^2 T(X,t)}{\partial t^2} + \frac{\tau_q^2}{2}\frac{\partial^3 T(X,t)}{\partial t^3}\Big] &= \nabla[\lambda\nabla T(X,t)] + \tau_T \frac{\partial\{\nabla[\lambda\nabla T(X,t)]\}}{\partial t} \\
+ \frac{\tau_T^2}{2}\frac{\partial^2\{\nabla[\lambda\nabla T(X,t)]\}}{\partial t^2} + Q(X,t) + \tau_q\frac{\partial Q(X,t)}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2 Q(X,t)}{\partial t^2}
\end{aligned}
\tag{2.9}
$$

Additionally, for $\lambda =$ const the last equation takes form

$$
\begin{aligned}
c\Big[\frac{\partial T(X,t)}{\partial t} + \tau_q\frac{\partial^2 T(X,t)}{\partial t^2} + \frac{\tau_q^2}{2}\frac{\partial^3 T(X,t)}{\partial t^3}\Big] &= \lambda\nabla^2 T(X,t) + \lambda\tau_T\frac{\partial[\nabla^2 T(X,t)]}{\partial t} \\
+ \lambda\frac{\tau_T^2}{2}\frac{\partial^2[\nabla^2 T(X,t)]}{\partial t^2} + Q(X,t) + \tau_q\frac{\partial Q(X,t)}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2 Q(X,t)}{\partial t^2}
\end{aligned}
\tag{2.10}
$$

As previously mentioned, dual phase lag equation (2.10) is often simplified by omitting appropriate components. For example, in several works (e.g. Tzou, 1995) the second order Taylor expression of heat flux and the first order Taylor expression of the temperature gradient are applied to take into account the phase lagging behavior. Ignoring the inner heat source (as in Tzou, 1995), the governing equation of temperature based on the DPL model is the following

$$
c\Big[\frac{\partial T(X,t)}{\partial t} + \tau_q\frac{\partial^2 T(X,t)}{\partial t^2} + \frac{\tau_q^2}{2}\frac{\partial^3 T(X,t)}{\partial t^3}\Big] = \lambda\nabla^2 T(X,t) + \lambda\tau_T\frac{\partial[\nabla^2 T(X,t)]}{\partial t}
\tag{2.11}
$$

It is also possible to consider the energy equation in the form (assuming that $Q(X,t) = 0$)

$$
c\Big[\frac{\partial T(X,t)}{\partial t} + \tau_q\frac{\partial^2 T(X,t)}{\partial t^2}\Big] = \lambda\nabla^2 T(X,t) + \lambda\tau_T\frac{\partial[\nabla^2 T(X,t)]}{\partial t} + \lambda\frac{\tau_T^2}{2}\frac{\partial^2[\nabla^2 T(X,t)]}{\partial t^2}
\tag{2.12}
$$

The most popular DPLE results from the assumption that the first-order approximation of formula (2.2) is used, and then (e.g. Tang and Araki, 1999; Al-Nimr *et al.*, 2004; Majchrzak and Mochnacki, 2014)

$$
c\Big[\frac{\partial T(X,t)}{\partial t} + \tau_q\frac{\partial^2 T(X,t)}{\partial t^2}\Big] = \lambda\nabla^2 T(X,t) + \lambda\tau_T\frac{\partial[\nabla^2 T(X,t)]}{\partial t} + Q(X,t) + \tau_q\frac{\partial Q(X,t)}{\partial t}
\tag{2.13}
$$

One can see that for $\tau_T = 0$, DPLE (2.13) takes form of the Cattaneo-Vernotte equation, while for $\tau_q = 0$ and $\tau_T = 0$ the well known macroscopic Fourier equation is obtained.

Taking into account the numerical examples presented in the final part of the paper, a modified form of the Neumann boundary condition must still be formulated, namely

$$\begin{aligned} q_b(X,t) + \tau_q \frac{\partial q_b(X,t)}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2 q_b(X,t)}{\partial t^2} \\ = -\lambda\Big[\mathbf{n}\cdot\nabla T(X,t) + \tau_T \frac{\partial[\mathbf{n}\cdot\nabla T(X,t)]}{\partial t} + \frac{\tau_T^2}{2}\frac{\partial^2[\mathbf{n}\cdot\nabla T(X,t)]}{\partial t^2}\Big] \end{aligned} \tag{2.14}$$

where $\mathbf{n}\cdot\nabla T(X,t)$ denotes normal derivative and $q_b(X,t)$ is the known boundary heat flux. In the case of simplified forms of the DPLE, the appropriate components in condition (2.14) should be neglected.

3. Formulation of the problem

Thermal processes proceeding in a thin metal film subjected to laser pulse are considered. A 1D problem is analyzed (heat transfer in the direction perpendicular to the layer is taken into account). The front surface $x = 0$ is irradiated by a laser pulse and according to (Tang and Araki, 1999; Kaba and Dai, 2005), the conductional heat transfer in the domain considered can be modeled using the DPLE in which the volumetric heat source $Q(x,t)$ is introduced. At the same time, for $x = 0$ and $x = L$, the non-flux conditions should be assumed. The laser irradiation is described by the following source term

$$Q(x,t) = \sqrt{\frac{\beta}{\pi}}\frac{1-R}{t_p\delta} I_0 \exp\Big[-\frac{x}{\delta} - \beta\frac{(t-2t_p)^2}{t_p^2}\Big] \tag{3.1}$$

where I_0 is the laser intensity, t_p is the characteristic time of the laser pulse, δ is the optical penetration depth, R is the reflectivity of the irradiated surface, and $\beta = 4\ln 2$.

In the most general case, the following DPLE is considered::

— for $0 < x < L$

$$\begin{aligned} \frac{\partial T(x,t)}{\partial t} + \tau_q\frac{\partial^2 T(x,t)}{\partial t^2} + w_q\frac{\tau_q^2}{2}\frac{\partial^3 T(x,t)}{\partial t^3} = a\frac{\partial^2 T(x,t)}{\partial x^2} + a\tau_T\frac{\partial^3 T(x,t)}{\partial t\partial x^2} \\ + w_T a\frac{\tau_T^2}{2}\frac{\partial^4 T(x,t)}{\partial t^2\partial x^2} + \frac{1}{c}Q(x,t) + \frac{\tau_q}{c}\frac{\partial Q(x,t)}{\partial t} + w_q\frac{\tau_q^2}{2c}\frac{\partial^2 Q(x,t)}{\partial t^2} \end{aligned} \tag{3.2}$$

where $a = \lambda/c$ is the diffusion coefficient, w_T and w_q are bivalent parameters. Here $w_T = 1$ and $w_q = 1$. For the "simplified" forms of DPLE, they are equal to $(0,1)$, $(1,0)$ and $(0,0)$.

As previously mentioned, $q_b(0,t) = q_b(L,t) = 0$ and the appropriate boundary conditions are of the form (Eq. (2.14)):

— for $x = 0$

$$\frac{\partial T(x,t)}{\partial x} + \tau_T\frac{\partial^2 T(x,t)}{\partial t\partial x} + w_T\frac{\tau_T^2}{2}\frac{\partial^3 T(x,t)}{\partial t^2\partial x} = 0 \tag{3.3}$$

— for $x = L$

$$\frac{\partial T(x,t)}{\partial x} + \tau_T\frac{\partial^2 T(x,t)}{\partial t\partial x} + w_T\frac{\tau_T^2}{2}\frac{\partial^3 T(x,t)}{\partial t^2\partial x} = 0 \tag{3.4}$$

The initial condition is also given for $t = 0$

$$T(x,0) = T_p \qquad \frac{\partial T(x,t)}{\partial t}\Big|_{t=0} = u(x) \qquad \frac{\partial^2 T(x,t)}{\partial t^2}\Big|_{t=0} = v(x) \tag{3.5}$$

where T_p is the initial temperature, while $u(x)$ and $v(x)$ are known functions.

4. Numerical algorithm

The algorithm presented below is based on the implicit scheme of the finite difference method (FDM).

Let $T_i^f = T(x_i, f\Delta t)$, where Δt is the time step, $x_i = ih$ (h is the geometrical mesh step) and $f = 0, 1, \ldots, F$. Taking into account initial conditions (3.5), on the assumption that $u(x) = v(x) = 0$, one has $T_i^0 = T_i^1 = T_i^2 = T_p$. For the transition $t^{f-1} \to t^f$ ($f \geqslant 3$), the approximate form of equation (3.2) resulting from the introduction of adequate differential quotients is as follows

$$
\begin{aligned}
&\frac{T_i^f - T_i^{f-1}}{\Delta t} + \tau_q \frac{T_i^f - 2T_i^{f-1} + T_i^{f-2}}{(\Delta t)^2} + w_q \frac{\tau_q^2}{2} \frac{T_i^f - 3T_i^{f-1} + 3T_i^{f-2} - T_i^{f-3}}{(\Delta t)^3} \\
&\quad = a \frac{T_{i-1}^f - 2T_i^f + T_{i+1}^f}{h^2} + \frac{a\tau_T}{\Delta t} \Big(\frac{T_{i-1}^f - 2T_i^f + T_{i+1}^f}{h^2} - \frac{T_{i-1}^{f-1} - 2T_i^{f-1} + T_{i+1}^{f-1}}{h^2} \Big) \\
&\quad + w_T \frac{a\tau_T^2}{2(\Delta t)^2} \Big(\frac{T_{i-1}^f - 2T_i^f + T_{i+1}^f}{h^2} - 2\frac{T_{i-1}^{f-1} - 2T_i^{f-1} + T_{i+1}^{f-1}}{h^2} + \frac{T_{i-1}^{f-2} - 2T_i^{f-2} + T_{i+1}^{f-2}}{h^2} \Big) \\
&\quad + \frac{1}{c} Q_i^f + \frac{\tau_q}{c} \Big(\frac{\partial Q}{\partial t} \Big)_i^f + w_q \frac{\tau_q^2}{2c} \Big(\frac{\partial^2 Q}{\partial t^2} \Big)_i^f
\end{aligned}
\tag{4.1}
$$

After mathematical transformations, one has

$$
\begin{aligned}
&- \frac{a[2(\Delta t)^2 + 2\tau_T \Delta t + w_T \tau_T^2]}{2h^2(\Delta t)^2} T_{i-1}^f + \Big[\frac{2(\Delta t)^2 + 2\tau_q \Delta t + w_q \tau_q^2}{2(\Delta t)^3} \\
&\quad + \frac{2a[2(\Delta t)^2 + 2\tau_T \Delta t + w_T \tau_T^2]}{2h^2(\Delta t)^2} \Big] T_i^f - \frac{a[2(\Delta t)^2 + 2\tau_T \Delta t + w_T \tau_T^2]}{2h^2(\Delta t)^2} T_{i+1}^f \\
&\quad = \frac{2(\Delta t)^2 + 4\tau_q \Delta t + 3w_q \tau_q^2}{2(\Delta t)^3} T_i^{f-1} - \frac{2\tau_q \Delta t + 3w_q \tau_q^2}{2(\Delta t)^3} T_i^{f-2} \\
&\quad + \frac{w_q \tau_q^2}{2(\Delta t)^3} T_i^{f-3} - \frac{a\tau_T(\Delta t + w_T \tau_T)}{h^2(\Delta t)^2} (T_{i-1}^{f-1} - 2T_i^{f-1} + T_{i+1}^{f-1}) \\
&\quad + \frac{a w_T \tau_T^2}{2h^2(\Delta t)^2} (T_{i-1}^{f-2} - 2T_i^{f-2} + T_{i+1}^{f-2}) + \frac{1}{c} Q_i^f + \frac{\tau_q}{c} \Big(\frac{\partial Q}{\partial t} \Big)_i^f + w_q \frac{\tau_q^2}{2c} \Big(\frac{\partial^2 Q}{\partial t^2} \Big)_i^f
\end{aligned}
\tag{4.2}
$$

Denoting

$$
\begin{aligned}
&A = -\frac{a[2(\Delta t)^2 + 2\tau_T \Delta t + w_T \tau_T^2]}{2h^2(\Delta t)^2} \qquad B = \frac{2(\Delta t)^2 + 2\tau_q \Delta t + w_q \tau_q^2}{2(\Delta t)^3} - 2A \\
&C_i^f = \frac{2(\Delta t)^2 + 4\tau_q \Delta t + 3w_q \tau_q^2}{2(\Delta t)^3} T_i^{f-1} - \frac{2\tau_q \Delta t + 3w_q \tau_q^2}{2(\Delta t)^3} T_i^{f-2} + \frac{w_q \tau_q^2}{2(\Delta t)^3} T_i^{f-3} \\
&\quad - \frac{a\tau_T(\Delta t + w_T \tau_T)}{h^2(\Delta t)^2} (T_{i-1}^{f-1} - 2T_i^{f-1} + T_{i+1}^{f-1}) + \frac{a w_T \tau_T^2}{2h^2(\Delta t)^2} (T_{i-1}^{f-2} - 2T_i^{f-2} + T_{i+1}^{f-2}) \\
&\quad + \frac{1}{c} Q_i^f + \frac{\tau_q}{c} \Big(\frac{\partial Q}{\partial t} \Big)_i^f + w_q \frac{\tau_q^2}{2c} \Big(\frac{\partial^2 Q}{\partial t^2} \Big)_i^f
\end{aligned}
\tag{4.3}
$$

one obtains

$$
A T_{i-1}^f + B T_i^f + A T_{i+1}^f = C_i^f \tag{4.4}
$$

The FDM equation resulting from the boundary condition for $x = 0$ is of the form

$$
\begin{aligned}
&\frac{T_1^f - T_0^f}{h} + \frac{\tau_T}{\Delta t} \Big(\frac{T_1^f - T_0^f}{h} - \frac{T_1^{f-1} - T_0^{f-1}}{h} \Big) \\
&\quad + \frac{w_T \tau_T^2}{2(\Delta t)^2} \Big(\frac{T_1^f - T_0^f}{h} - 2\frac{T_1^{f-1} - T_0^{f-1}}{h} + \frac{T_1^{f-2} - T_0^{f-2}}{h} \Big) = 0
\end{aligned}
\tag{4.5}
$$

or

$$
\begin{aligned}
&-[2(\Delta t)^2 + 2\tau_T \Delta t + w_T \tau_T^2] T_0^f + [2(\Delta t)^2 + 2\tau_T \Delta t + w_T \tau_T^2] T_1^f \\
&\quad = (2\tau_T \Delta t + 2w_T \tau_T^2)(T_1^{f-1} - T_0^{f-1}) - w_T \tau_T^2 (T_1^{f-2} - T_0^{f-2})
\end{aligned}
\tag{4.6}
$$

Let us denote

$$
D = 2(\Delta t)^2 + 2\tau_T \Delta t + w_T \tau_T^2 \qquad E = 2\tau_T \Delta t + 2w_T \tau_T^2 \tag{4.7}
$$

then

$$
-DT_0^f + DT_1^f = E(T_1^{f-1} - T_0^{f-1}) - w_T \tau_T^2 (T_1^{f-2} - T_0^{f-2}) \tag{4.8}
$$

In a similar way, for $x = L$, one has

$$
-DT_{n-1}^f + DT_n^f = E(T_n^{f-1} - T_{n-1}^{f-1}) - w_T \tau_T^2 (T_n^{f-2} - T_{n-1}^{f-2}) \tag{4.9}
$$

So, the final form of the system of equations corresponding to the transition $t^{f-1} \to t^f$ $(f \geqslant 3)$ is the following

$$
\begin{aligned}
&-DT_0^f + DT_1^f = E(T_1^{f-1} - T_0^{f-1}) - w_T \tau_T^2 (T_1^{f-2} - T_0^{f-2}) \\
&AT_{i-1}^f + BT_i^f + AT_{i+1}^f = C_i^f \qquad i = 1, 2, \ldots, n-1 \\
&-DT_{n-1}^f + DT_n^f = E(T_n^{f-1} - T_{n-1}^{f-1}) - w_T \tau_T^2 (T_n^{f-2} - T_{n-1}^{f-2})
\end{aligned}
\tag{4.10}
$$

So, the transition from t^{f-1} to t^f $(f \geqslant 3)$ requires solving of the system of equations with a three-band main matrix which is the fastest solved using the Thomas algorithm.

5. Examples of computations

Thin metal films ($L = 100\,\mathrm{nm}$) made of chromium, nickel and gold have been considered. The surface $x = 0$ of the domain is subjected to the laser pulse. The parameters determining the capacity of the internal heat source (Eq. (3.1)) are equal to $I_0 = 13.7\,\mathrm{J/m^2}$, $t_p = 0.1\,\mathrm{ps}$, $\delta = 15.3\,\mathrm{nm}$, $R = 0.93$. The initial temperature of the domain equals $T_p = 300\,\mathrm{K}$, while the initial values of functions are $u(x) = 0$, $v(x) = 0$. Differential mesh parameters are $n = 1000$, $\Delta t = 0.0001\,\mathrm{ps}$.

At the stage of numerical computations, constant values of thermophysical parameters have been assumed (mainly due to lack of other data in the literature) – see Table 1.

Table 1. Thermophysical parameters (Tzou, 2015)

	Chromium	Gold	Nickel
c [MJ/(m^{3}K)]	3.21484	2.4897	4
λ [W/(mK)]	93	315	90.8
τ_q [ps]	0.136	8.5	0.82
τ_T [ps]	7.86	90	10

Computations have been performed in versions corresponding to $w_T = 0$, $w_q = 0$ (first-order DPLE), $w_T = 1$, $w_q = 1$ (second-order DPLE), $w_T = 0$, $w_q = 1$ and $w_T = 1$, $w_q = 0$ (mixed order DPLE). Additionally, for comparative purposes, numerical solutions of the classical Fourier problems have been also found. The results are presented in the form of heating/cooling curves at the irradiated surface. The set of solutions for the chromium layer is shown in Fig. 1. For the other materials (Figs. 2 and 3), the solutions corresponding to the Fourier model, $w_T = 0$, $w_q = 0$ and $w_T = 1$, $w_q = 1$ are distinguished. The discussion of the results obtained will be carried out in the next Section.

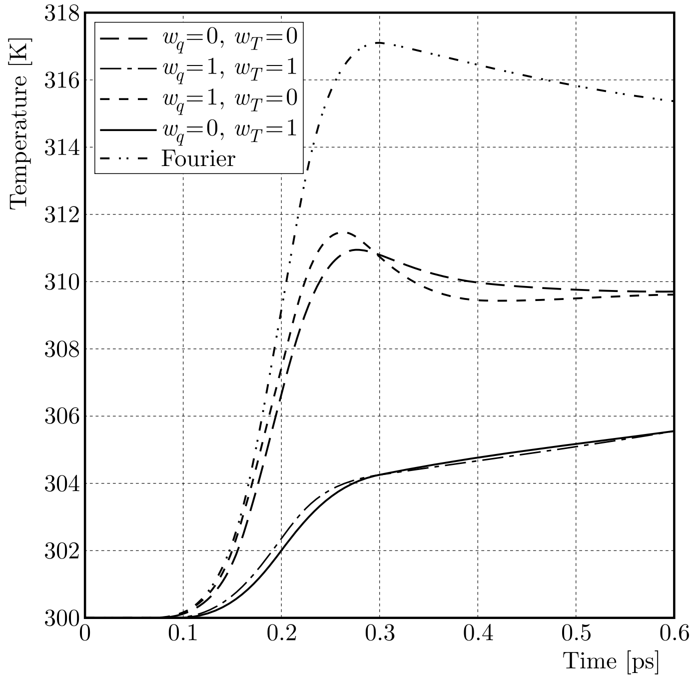

Fig. 1. Temperature history at the irradiated surface for different models (chromium)

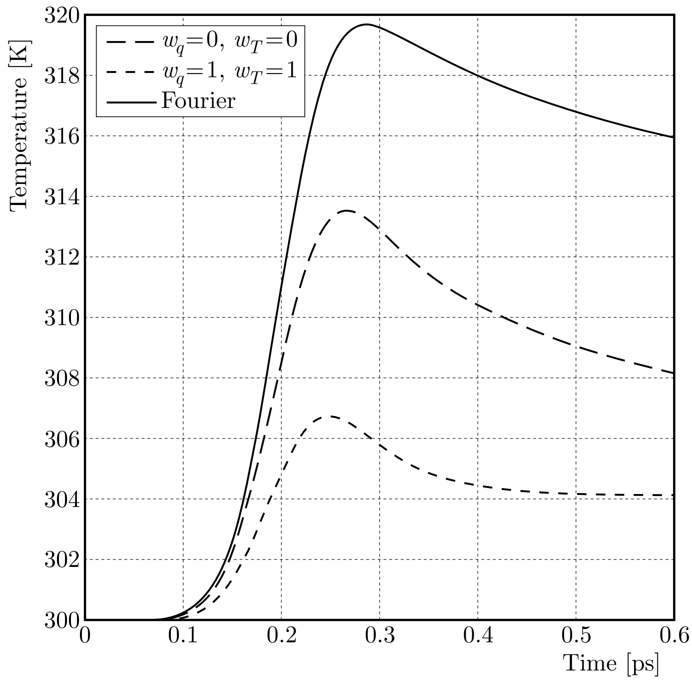

Fig. 2. Temperature history at the irradiated surface for different models (gold)

6. Conclusions

Different (in the sense of the order) models using the dual phase lag equation give different results. Here, one can see some regularities. In relation to the model based on the second-order DPLE, the solution resulting from the first-order equation is clearly overstated. This is the case for all the materials in question. The fact that the Fourier model gives a solution over DPLE has been repeatedly confirmed in numerous papers. This is a natural consequence of the delay

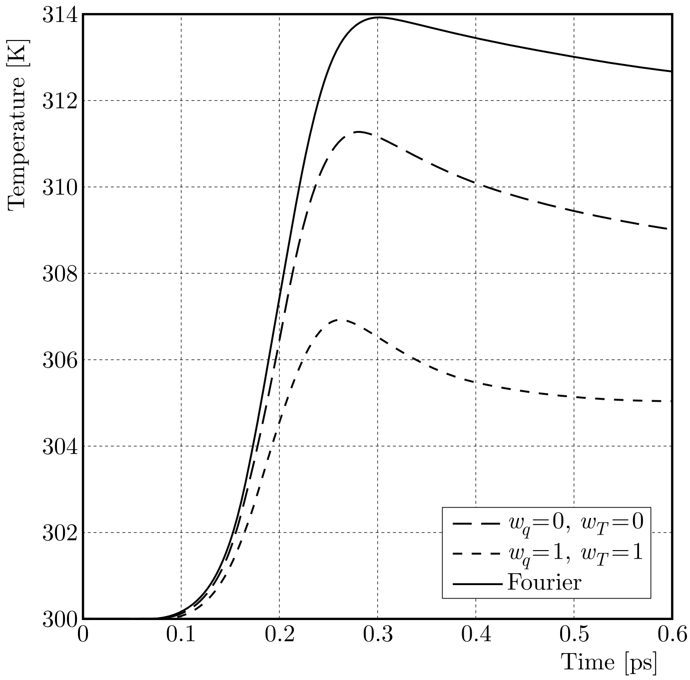

Fig. 3. Temperature history at the irradiated surface for different models (nickel)

times introduced. In the case of mixed models, the omission of the component containing τ_T^2 (Eq. (2.11)) leads to results close to the solution of the first-order DPLE – see Fig. 1. On the other hand, the omission of the component containing τ_q^2 (Eq. (2.12)) gives a solution similar to the solution of the second-order DPLE. The same trend is observed for the remaining materials. This results from the much larger (in the case of metals) value of the thermalization time versus the relaxation one. Therefore, more components of the Taylor series should be included on the right hand side of the generalized Fourier law. Summing up, the problems connected with the modeling of thermal processes in metal microdomains should be solved using the second-order dual phase lag equation. If the delay times vary less, then the solution based on the first-order model is sufficiently accurate.

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